



MONTCLAIR STATE
UNIVERSITY

Montclair State University
**Montclair State University Digital
Commons**

Theses, Dissertations and Culminating Projects

5-2012

Enumeration of Independent Sets in Graphs

James Alexander
Montclair State University

Follow this and additional works at: <https://digitalcommons.montclair.edu/etd>



Part of the [Mathematics Commons](#)

Recommended Citation

Alexander, James, "Enumeration of Independent Sets in Graphs" (2012). *Theses, Dissertations and Culminating Projects*. 714.

<https://digitalcommons.montclair.edu/etd/714>

This Thesis is brought to you for free and open access by Montclair State University Digital Commons. It has been accepted for inclusion in Theses, Dissertations and Culminating Projects by an authorized administrator of Montclair State University Digital Commons. For more information, please contact digitalcommons@montclair.edu.

MONTCLAIR STATE UNIVERSITY

/ Enumeration of Independent Sets in Graphs /

by

James Alexander

A Master's Thesis Submitted to the Faculty of

Montclair State University

In Partial Fulfillment of the Requirements

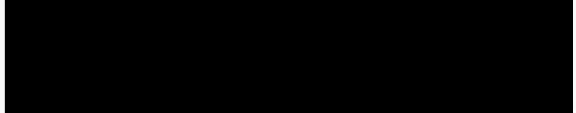
For the Degree of

Master of Science

May 2012

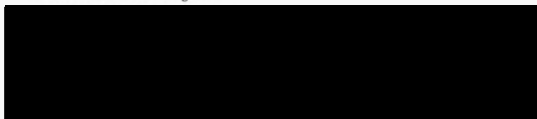
College of Science and Mathematics
Department of Mathematical Sciences

Thesis Committee: 

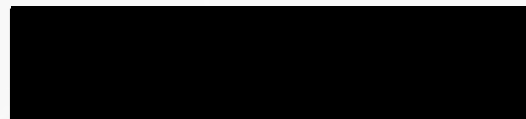


Dr. Jonathan Cutler
Thesis Sponsor

Certified by:

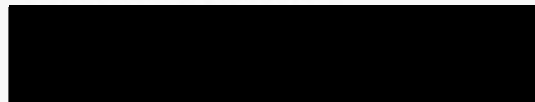


Dr. Robert Prezant
Dean of CSAM



Dr. Aihua Li
Committee Member

May 3, 2012
Date



Dr. Diana Thomas
Committee Member



Dr. Helen Roberts
Department Chair

Abstract

An independent set is one of the most natural structures in a graph to focus on, from both a pure and applied perspective. In the realm of graph theory, and any concept it can represent, an independent set is the mathematical way of capturing a set of objects, none of which are related to each other. As graph theory grows, many questions about independent sets are being asked and answered, many of which are concerned with the enumeration of independent sets in graphs. We provide a detailed introduction to general graph theory for those who are not familiar with the subject, and then develop the basic language and notation of independent set theory before cataloging some of the history and major results of the field. We focus particularly on the enumeration of independent sets in various classes of graphs, with the heaviest focus on those defined by maximum and minimum degree restrictions. We provide a brief, specific history of this topic, and present some original results in this area. We then speak about some questions which remain open, and end the work with a conjecture for which we provide strong, original evidence. In the appendices, we cover all other necessary prerequisites for those without a mathematical background.

Enumeration of Independent Sets in Graphs

A THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science

by

James Alexander
Montclair State University
Montclair, NJ
2012

Contents

List of Figures	iii
1 Introduction	1
1.1 Basic graph theory definitions and notation	1
1.2 Independent set definitions and notation	9
2 On the total numbers of independent sets in graphs	10
2.1 Initial Observations and Specific Counts	10
2.2 Bounding the quantity $i(G)$	13
3 Independent sets of a fixed size	17
3.1 On the study of $i_t(G)$	17
3.2 Independent sets of maximum size	19
3.3 Independent sets of various sizes	21
4 On the independence polynomial of a graph	24
4.1 Important observations and a brief history	25
4.2 The Roller-Coaster Conjecture	29
5 The imposition of degree restrictions	31
5.1 The history of degree restricted study	31
5.2 On a 2011 conjecture	33
5.3 A stronger conjecture	35
6 Original results	37
6.1 Our main original result	37
6.2 On the sharpness of our main result	46
6.3 Corollaries of our main result	48
6.4 Other original results	50
7 Implications of our original results	52
7.1 Relationships to work of the past	52
7.2 A justified conjecture	54
Bibliography	54

Appendices	56
A Basic set theory	57
A.1 Definition of a set and explanation of the definition of a graph	57
A.2 How to draw a graph	58
A.3 Related notation and basic set operations	59
A.4 Indexing	61
B Mathematical proofs	64
B.1 Introduction to mathematical proof	64
B.2 Combinatorial Proofs	65
B.3 Proof by Contradiction	66
B.4 Proof by Induction	67
C Introduction to basic combinatorics	70
C.1 A brief introduction to enumerative combinatorics	70
C.2 Counting Subgraphs	73
C.3 Unimodality of binomial coefficients	74
C.4 A significant result	75

List of Figures

1.1	Three trees on five vertices	5
1.2	A bipartite graph on nine vertices	6
1.3	The complete bipartite graph $K_{3,3}$	7
1.4	A star on five vertices	7
1.5	An independent set of size 3	9
5.1	The cycle C_5 and the complete bipartite graph $K_{2,3}$	35
A.1	The path P_4	59
A.2	The complete graph K_4 and the empty graph E_4	60
A.3	A bipartite graph on six vertices	63

Chapter 1

Introduction

1.1 Basic graph theory definitions and notation

This section provides a brief introduction to graph theory. Those familiar with the subject should skip this section, and possibly Section 1.2, only referring back if they are not familiar with the notation being used. For even the most basic definitions, some familiarity with basic set theory is necessary, and so those who are unfamiliar with basic set theory should see Appendix A for the necessary prerequisites. Those without a mathematical background should read all appendices entirely before proceeding. Graph theory is a very visual subject, and so we recommend always having a pen and paper around, and drawing plenty of pictures for clarity. In Appendix A, we describe how to draw a graph.

Basic Definitions

A *graph* G is a mathematical structure consisting of a finite nonempty set $V(G)$ of objects called *vertices*, and a set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. If $\{u, v\} \in E(G)$, we say that u and v are *adjacent*, and refer to $\{u, v\}$ as an edge between u and v . For convenience, we denote edge $\{u, v\}$ by uv . We say that uv is the *incident* to u and v . We notice that graphs cannot have edges of the form vv , or without multiple edges between vertices (as shown in Proposition B.3.1 of Appendix B). It is common to denote the number of vertices of G , referred to as the *order* of G , by $n(G) := |V(G)|$, and to denote the number of edges of G by $e(G) := |E(G)|$. We call a graph *complete* if it has all possible edges, and *empty* if it has none. We denote the complete graph on n vertices K_n , and the empty graph on n vertices E_n .

Degree of a Vertex

For any graph G and $v \in V(G)$, we call $N_G(v) := \{u \in V(G) : uv \in E(G)\}$ the *neighborhood of v* , and $d_G(v) := |N_G(v)|$ the *degree of v* . That is, we call the number of edges incident to a vertex v the degree of v , and we call all the vertices that lie on the other ends of those edges the neighbors of v . Let us look at the following result, which is often the first theorem covered in an introductory graph theory course. Though fairly easy to prove, it displays some very useful ideas that we will need throughout this book. This theorem usually goes by one of two names. The first is the degree-sum formula, the second is the one we will refer to it by here. The proof provided is a concise combinatorial proof, which is explained in detail in Appendix B.

Theorem 1.1.1 (The First Theorem of Graph Theory). *For any graph G ,*

$$\sum_{v \in V(G)} d_G(v) = 2e(G).$$

Proof. Our goal is to show that when you add up the degrees of every vertex in the graph, this counts the number of edges twice. This is the case, for when we are summing the degrees of the vertices of G , we count each edge twice, one for each vertex incident to it. \square

This theorem establishes an important relationship between the degrees of the vertices of a graph, and the number of edges in that graph. As we would expect, since K_n is the graph with all possible edges, this theorem tells us that the sum of the degrees of all vertices of K_n is greater than the sum of the degrees of the vertices for any other n -vertex graph (graph on n vertices). Similarly, this theorem tells us that the sum of the degrees of the vertices of E_n is 0, the smallest possible degree sum of any graph.

We notice that this theorem does not directly deal with how large or small the degrees of specific vertices are. For these types of questions, we need some additional notation. We denote the smallest degree of any vertex in a graph G , referred to as the *minimum degree* of G , by $\delta(G) := \min_{v \in V(G)} \{d_G(v)\}$. Similarly, we denote *maximum degree* of G by $\Delta(G) := \max_{v \in V(G)} \{d_G(v)\}$. For any non-negative integer r , we say that G is *r -regular* if $\delta(G) = \Delta(G) = r$, that is, every vertex of G has degree r . Though we will not explore any results about maximum and minimum degree just yet, there will be plenty of these types of results to come.

Walks

For any graph G and $u, v \in V(G)$, we define a *u, v -walk* as a alternating sequence of vertices and edges in G beginning with u and ending with v such that each edge joins the vertices which precede and follow it. Naturally, we call the number of edges of a walk the *length* of that walk. We call a *u, v -walk* which does not repeat a vertex

a u, v -path, and a u, v -walk which does not repeat an edge a u, v -trail. We use the notation P_n to denote a path on n vertices. Specifically, when we say that we are considering the path P_n on vertices labeled v_1, v_2, \dots, v_n , we are considering the graph with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. We will use this specific definition (based on this labeling) many times in coming sections.

A *circuit* is defined as a trail of length at least 3 which begins and ends on the same vertex, and a circuit which repeats no vertex other than the first/last is called a *cycle*. We use the notation C_n to denote a cycle on n vertices, for $n \geq 3$. We require $n \geq 3$, for it is impossible to construct a cycle on less than this many vertices. We call a cycle an *odd cycle* if it contains an odd number of edges, and an *even cycle* if it contains an even number. Similarly to how we defined P_n in terms of a labeling, when we say that we are considering the cycle C_n on vertices v_1, v_2, \dots, v_n , this is a convenient way of saying that we are considering the graph with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$.

When proving things about paths and cycles, it is very common to use the proof technique, induction. We introduce this technique in Appendix B, and provide a proof that, for any $n \geq 1$, $e(P_n) = n - 1$. This provides a very good warm-up for induction proofs to come, and provides us with this property, which will prove helpful in coming sections.

Subgraphs

It is very common to talk about paths and cycles, but more so as parts of larger graphs than as graphs themselves. Some reasons why will be discussed in coming sections. When we say that we look at paths and cycles as parts of bigger graphs, this is a fairly ambiguous statement. In order to make this statement more concrete, we need the following definition.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

So, the preceding lines can be restated as, we often look at paths and cycles as subgraphs of other graphs (with a deeper explanation of this offered in Appendix C, subsection: Counting subgraphs). We provide some convenient notation for denoting some of the more common types of subgraphs in the following paragraphs.

When doing graph theory proofs, it is sometimes convenient to have a way to notate a given graph less a specified vertex and all edges which are adjacent to it. For a graph G , and a given vertex v , we use $G - \{v\}$ to denote the subgraph of G obtained by removing vertex v and all edges incident to v . We extend this notation to more than one vertex by, for any vertices $v_1, v_2, \dots, v_m \in V(G)$, letting $G - \{v_1, v_2, \dots, v_m\}$ denote the subgraph of G obtained by removing vertices v_1, v_2, \dots, v_m and all edges incident to these vertices. On the other hand, for any vertices $v_1, v_2, \dots, v_m \in V(G)$,

we define the *subgraph of G induced by vertices v_1, v_2, \dots, v_m* , denoted $G[v_1, v_2, \dots, v_m]$, to be the subgraph of G made up of the vertices v_1, v_2, \dots, v_m and all edges between them. For a some specific example, see Appendix C.2.

Connectivity

We would, intuitively, want use the word connected to describe a graph for which there is a walk between any two vertices, that is, one which is a single piece, that you can trace (along edges) without picking up your pen (such as the graph G in Example A.2.1 of Appendix A). It turns out that we call a graph *connected*, if there is a path between any two vertices of the graph, but that these definitions are actually equivalent. To see this, consider the following theorem, which we prove by induction.

Theorem 1.1.2. *For any graph G , and any $u, v \in V(G)$, if there is a u, v -walk in G , then there is a u, v -path in G .*

Proof. We show that, in fact, every u, v -walk contains a u, v -path. That is, if there is a u, v -walk in G , then there is not just a u, v -path, but we can find the path defined within the sequence defining the walk. We do so by induction on the length, say l , of our given u, v -walk, to show that it is true for walks of any length 0 or greater. If $l = 0$, then our u, v -walk contains no edges, and so the walk consists of a single vertex. That is, $u = v$. This vertex by itself is a u, v -path of length 0 as nothing is repeated, and so the assertion immediately holds in this case.

Now assume that the assertion holds for all walks of length $l = k$ or less, and assume that there is a u, v -walk of length $k + 1$. If this u, v -walk is a u, v -path, then we are finished, so assume that it is not. That is, that the u, v -walk does have some repeated vertex, say $w \in V(G)$. Then, deleting the edges and vertices between appearances of w (in the sequence that is the walk), leaving one copy of w , yields a shorter u, v -walk, with length less than k . By our induction assumption, there is a u, v -path in G , as desired. \square

We call the maximal connected subgraphs of a graph G its *components*. Informally, the components of a graph are the connected pieces. If G is connected, it has one component. If G is not connected, but can be drawn to look like two connected graphs placed next to each other, then it has two components. In general, if it is essentially k different connected (disjoint) graphs, for some positive integer k , then it has k components.

Trees

A connected graph with no cycles is called a *tree*. If one draws a few connected graphs with no cycles, it is easy to see why this type of graph is called a tree. No matter

how complex it is drawn, it always branches off in a way that looks tree-like. Some small examples of trees can be found in Figure 1.1.

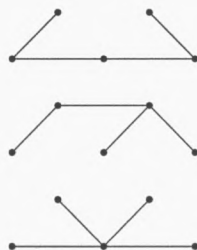


Figure 1.1: Three trees on five vertices

When drawing trees, one should notice unavoidable vertices of degree one that lie at the end of these branches. The fact that every tree has at least one of these vertices of degree one turns out to be very useful when proving things about trees, and so we state and prove this here, formally, and refer back to it in coming sections. This will be a proof by contradiction.

Theorem 1.1.3. *Every tree with at least two vertices contains a vertex of degree one.*

Proof. Let T be a tree with $n(T) \geq 2$. Then, as T is connected, it must contain an edge, so it must contain at least one path. Consider the longest path in T , made up of vertices, say, $v_1, v_2, \dots, v_k \in V(T)$, and edges, say, $v_1v_2, v_2v_3, \dots, v_{k-1}v_k \in E(T)$. We claim that v_1 is a vertex of degree one, i.e., that v_2 is its only neighbor. We show this by contradiction. That is, we assume, to the contrary, that v_1 has some other neighbor $v \in V(T)$ such that $v \neq v_2$. If v lies on the path, i.e., if $v = v_i$ for some $i \in \{3, \dots, k\}$, then we can form a cycle with the edge v_1v_i and the part of the path from v_1 to v_i , which is a contradiction since trees have no cycles by definition. If v does not lie on the path, then the given path together with v and the edge v_1v is a longer path, contradiction our assumption that our longest path in T . In either case we obtain a contradiction, and thus v_1 must not be adjacent to any vertex other than v_2 . The vertex v_1 of T has degree one. \square

We notice that the proof of this actually implies that every tree with at least two vertices contains at least two vertices of degree one, since the vertex v_k in this proof surely has to have degree one as well. If G is not necessarily connected, but all components of G are trees, G is called a *forest*. Since every tree which contains an edge (and thus contains at least two vertices) contains a vertex of degree one, every forest which is not E_n surely contains a vertex of degree one. We will use this fact as well.

Bipartite Graphs

We call a graph G *bipartite* if $V(G)$ can be partitioned into 2 sets, say A and B , such that no two vertices of A are adjacent, and no two vertices of B are adjacent. Sets of

these type, in which there are no edges, are called independent sets, and they are the primary focus of this work. We will define them formally in the next section, but we would like to mention now that a bipartite graph is exactly a graph whose vertex set can be partitioned into two of these, so called, independent sets. Figure 1.2 depicts a bipartite graph on nine vertices.



Figure 1.2: A bipartite graph on nine vertices

If G is a bipartite graph, and $A \cup B$ be a partitioning of $V(G)$, then we call A and B the *partite* sets of G and say that G has *bipartition* $V(G) = A \cup B$. Bipartite graphs will come up fairly often in this book, and so we package some initial observations about bipartite graphs that we will need into the following proposition. We recommend being comfortable with these before moving on.

Proposition 1.1.4. *If G is a bipartite graph with bipartition $V(G) = A \cup B$, then*

1. $n(G) = |A| + |B|$,
2. $e(G) = \sum_{v \in A} d_G(v) = \sum_{v \in B} d_G(v)$, and
3. For any $v \in A$, $d_G(v) \leq |B|$.

Proof. Since $V(G)$ is, by definition, partitioned into sets A and B , it immediately follows that $n(G) = |V(G)| = |A| + |B|$ from the definition of a partitioning, and so (1) is proven. To prove (2), notice that, by definition of bipartite, every edge must have one endpoint in A and one endpoint in B . Thus, if we simply count all the endpoints that lie, or equivalently, the sum of all the degrees of the vertices of A , or B , then we will have counted $e(G)$ exactly. This is exactly what (2) states. To prove (3), notice that, since vertices of A cannot be adjacent to other vertices in A , for any $v \in A$, all edges adjacent to v must also be adjacent to vertices of B . Since there are only $|B|$ vertices in B by definition, each v can be adjacent to at most $|B|$ vertices. This proves (3). Just as with (2), we could have equivalently stated (3) as, for any $w \in B$, $d_G(w) \leq |A|$. These types of symmetry will prove very important. \square

One very special type of bipartite graph, which will come up quite a bit, is what is called a *complete bipartite graph*. Recall that we use the word complete to mean

all possible edges, and the complete bipartite graph is exactly this: the bipartite graph with all possible edges. Since, by definition of bipartite, we can have no edges within partite sets, this graph is the bipartite graph where a vertex in one partite set is adjacent to every vertex in the other partite set. If G is a complete bipartite graph with bipartition $V(G) = A \cup B$, we denote G by $K_{|A|,|B|}$. Figure 1.3 depicts the complete bipartite graph $K_{3,3}$.

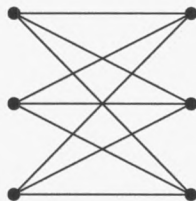


Figure 1.3: The complete bipartite graph $K_{3,3}$

Another example of a complete bipartite graph on n vertices with partite sets of size 1 and $n - 1$, regardless of what we call those partite sets, we would label this complete bipartite graph $K_{1,n-1}$. It turns out that this particular complete bipartite graph is called a *star*. One example of a star is Figure 1.4.



Figure 1.4: A star on five vertices

We make some observations about these types of bipartite graphs.

Proposition 1.1.5. *If G is a complete bipartite graph with bipartition $V(G) = A \cup B$, that is, if $G = K_{|A|,|B|}$, then*

1. *For any $v \in A$, $\deg_G(v) = |B|$ and*
2. *$e(G) = |A| \cdot |B|$.*

Proof. (1) follows directly from the definition of complete bipartite, as every vertex in A is adjacent to every vertex in B . To show (2), we use an argument similar to that used in Proposition 1.1.4. As discussed there, we can count $e(G)$ by counting the sum of the degrees of the vertices in A . For each $v \in A$, $d_G(v) = |B|$ by (1), and so we are adding $|B|$ once for each $v \in A$, and there are $|A|$ vertices in A . Formally,

$$e(G) = \sum_{v \in A} d_G(v) = \sum_{v \in A} |B| = |B| \sum_{v \in A} 1 = |B| \cdot |A|,$$

as desired.



End of Section Remarks

We have covered all the basic language and notation that we will need. For those not familiar with the subject, this very brief introduction may not have been completely satisfying. For this case, we recommend [1] for a more comprehensive introduction. Now that we have covered all these basic points, we move onto discussing independent sets in particular, in the next section.

1.2 Independent set definitions and notation

An *independent set* (of vertices) of a graph G is a subset I of $V(G)$ such that no two vertices of I are adjacent. For example, the gray vertices in Figure 1.5 form an independent set.

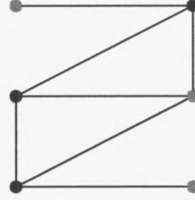


Figure 1.5: An independent set of size 3

An independent set is one of the most natural structures in a graph to focus on, from both a pure and applied perspective. In the realm of graph theory, and any concept it can represent, an independent set is the mathematical way of capturing a set of objects, none of which are related to each other. The following symbols are many of the most common in independent set theory, and they apply to any graph G . We will use all of these throughout the coming sections, and so, for reference, we present them here in a convenient list.

- $\mathcal{I}_t(G)$ denotes the set of all independent sets in G of size t , for any $t \in \mathbb{N}$.
- $i_t(G)$ denotes $|\mathcal{I}_t(G)|$, the number of independent sets of size t in G , for any $t \in \mathbb{N}$.
- $\mathcal{I}(G)$ denotes $\bigcup_{t \in \mathbb{N}} \mathcal{I}_t(G)$, the set of all independent sets in G .
- $i(G)$ denotes $|\mathcal{I}(G)|$, the total number of independent sets in G .
- $\alpha(G)$ denotes the size of the largest independent set in G .

The quantity $\alpha(G)$ is most often referred to as the *independence number* of G , and $i(G)$ is usually not given a name, but when it is, it is most often called the Fibonacci number of G for reasons we will explain in the next section.

The final thing we must consider is what is called the *independence polynomial* of a graph G , denoted $P(G, x)$. It is the polynomial whose coefficients are the terms of the sequence $(i_t(G))_{t \in \mathbb{N}}$. That is,

$$P(G, x) := \sum_{t \in \mathbb{N}} i_t(G) x^t.$$

Note that $P(G, 1) = i(G)$. We will use observations such as these throughout the coming sections.

Chapter 2

On the total numbers of independent sets in graphs

Many questions about independent sets are being asked and answered, a large percentage of which are concerning the quantities $i(G)$ and $i_t(G)$. The quantity $i(G)$ has a very interesting history. Chapter 2.1 is dedicated to this history of $i(G)$, and to the initial related results, which dealt with specifically counting the number of independent sets in certain graphs. Once we have gone through these specific counts, we move on to Chapter 2.2 which deals with some more contemporary questions, and which explains some of the motivation behind studying the number of independent sets of a fixed size, that is, $i_t(G)$, which is the focus of Chapter 3.

2.1 Initial Observations and Specific Counts

The quantity $i(G)$ was first explicitly considered by Prodinger and Tichy in [2], who referred to it as the Fibonacci number of a graph. Before we can see why, in Proposition 2.1.1, we first offer the following definition of the Fibonacci numbers for the reader who is not familiar with this sequence of integers, or who may have an equivalent definition which involves slightly different indexing than the one we'll use here. The n th Fibonacci number, denoted F_n , is defined by

$$F_0 = F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

We proceed to the proof of Proposition 2.1.1 using this definition.

Proposition 2.1.1 (Prodinger and Tichy 1982). *For any positive integer n ,*

$$i(P_n) = F_{n+1}.$$

Proof. We show that $i(P_1) = F_2$ and $i(P_2) = F_3$, and then that $i(P_n) = i(P_{n-1}) + i(P_{n-2})$ for $n \geq 3$. That is, we show the result (inductively) by showing that $i(P_n)$

starts like the Fibonacci numbers, and grows like them for all n , proving that they coincide. As P_1 is made up of a single vertex, the only independent sets of P_1 are that vertex, and \emptyset , which is trivially independent as it contains no edges. Thus, $i(P_1) = 2 = F_2$. Now consider $i(P_2)$. The set $V(P_2)$ contains 2 vertices, say v_1 and v_2 , which are adjacent. It follows that the only independent sets of P_2 are $\{v_1\}$, $\{v_2\}$, and \emptyset . This, $i(P_2) = 3 = F_3$.

It remains to show that $i(P_n) = i(P_{n-1}) + i(P_{n-2})$ for $n \geq 3$. So, for any positive integer $n \geq 3$, consider the path P_n on vertices v_1, v_2, \dots, v_n . We enumerate $i(P_n)$ by partitioning the set of independent sets of P_n , $\mathcal{I}(P_n)$, into those independent sets that contain the vertex v_n , and those that do not.

First, consider all independent sets of P_n which don't contain the vertex v_n . This is exactly the number of independent sets formed in $P_n[v_1, v_2, \dots, v_{n-1}]$, which is a P_{n-1} . Thus, the number of independent sets of P_n which don't contain v_n is $i(P_{n-1})$. Now consider all independent sets which do contain v_n . As the vertex v_n is only adjacent to vertex v_{n-1} , it can be in independent sets with any vertex of $P_n[v_1, v_2, \dots, v_{n-2}]$. That is, all independent sets of P_n which contain v_n are exactly all sets of the form $I \cup \{v_n\}$ where I is any independent subset of $\{v_1, v_2, \dots, v_{n-2}\}$. There are $i(P_{n-2})$ such possible independent subsets as $P_n[v_1, v_2, \dots, v_{n-2}]$ is a P_{n-2} . Thus, the number of independent sets of P_n which do contain v_n is exactly $i(P_{n-2})$. Putting together everything that we have just observed, we have that

$$i(P_n) = \left(\begin{array}{c} \text{Number of independent} \\ \text{sets that contain } v_n \end{array} \right) + \left(\begin{array}{c} \text{Number of independent} \\ \text{sets that don't contain } v_n \end{array} \right) = i(P_{n-1}) + i(P_{n-2}),$$

as desired. □

The name Fibonacci number for $i(G)$ is still the most common name for the quantity throughout graph theory, but it is rarely used. In modern graph theory, the quantity is almost always left untitled. However, $i(G)$ does go by different names in different fields. For example, in molecular chemistry, $i(G)$ is almost always referred to as the Merrifield-Simons index of G . For some discussion on this title, and for some specific uses of this quantity in molecular chemistry, we can see [3].

Once $i(P_n)$ has been characterized for all positive integers n , it is natural to ask what the count would be for graphs of similar structure. One graph of incredibly similar structure is the cycle, as it is simply a path with the endpoints joined together. It turns out that a similar type of count does hold for $i(C_n)$, as explained by Prodinger and Tichy in the same paper, but not in terms of the Fibonacci numbers. This count comes in the form of the very similar Lucas numbers. The n th Lucas number, denoted L_n , is defined by

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2.$$

The Lucas numbers are closely related to the Fibonacci numbers, as is expected,

because of the similarity between paths and cycle. Just how related the sequences are can be seen in the following, which we will use when proving the next proposition which counts $i(C_n)$ explicitly in terms of the Fibonacci numbers.

Claim 2.1.1. For $n \geq 2$, $L_n = F_{n-1} + F_{n+1}$.

Proof. This is a quick proof by induction. The result surely holds for $n = 2$, as $L_2 = 3 = 1 + 2 = F_1 + F_3$. Assume that $L_n = F_{n-1} + F_{n+1}$ for integers between 2 and $n \geq 2$. Then, by simply expanding L_{n+1} by definition, and then applying this assumption, we have

$$L_{n+1} = L_n + L_{n-1} = F_{n-1} + F_{n+1} + F_{n-2} + F_n = (F_{n-1} + F_{n-2}) + (F_{n+1} + F_n) = F_n + F_{n+2},$$

as desired, with the last equality holding by definition of the Fibonacci numbers. \square

We now prove the proposition about $i(C_n)$.

Proposition 2.1.2 (Prodinger and Tichy 1982). *For any $n \geq 3$,*

$$i(C_n) = L_n$$

Proof. As mentioned in the introduction, we require $n \geq 3$ because it is impossible to create a cycle on less than 3 vertices. As C_3 is complete (meaning that every vertex is adjacent to every other vertex), the only independent sets of C_3 are the empty set and each single vertex. As there are 3 vertices, $i(C_3) = 4 = L_3$, as desired.

Now, consider C_n for $n \geq 4$, on vertices v_1, v_2, \dots, v_n . We directly show that $i(C_n) = L_n$ using an approach similar to that used in Proposition 2.1.1. We count the number of independent sets in C_n which do and do not contain the vertex v_1 . We note that the number of independent sets of C_n which do not include v_1 are exactly the independent sets of $C_n - \{v_1\}$. It is easy to see that $C_n - \{v_1\}$ is a path on $n - 1$ vertices, and so, by Proposition 2.1.1, $i(C_n - \{v_1\}) = F_{n+1}$. Similarly, the number of independent sets which do not contain v_1 is exactly the number of independent sets formed by the vertices to which v_1 is not adjacent. That is, $i(C_n - \{v_1, v_2, v_n\})$. Since $n \geq 4$, this is a path of length at least $n - 3 \geq 1$, and so $i(C_n - \{v_1, v_2, v_n\}) = F_{n-1}$. It follows that

$$i(C_n) = F_{n-1} + F_{n+1} = L_n,$$

as desired, with the last equality holding by Claim 2.1.1. \square

We opened this section by counting the number of independent sets in a particular type of tree, namely, the path. It will turn out, as shown in Theorem 2.2.2, that among all trees, paths have the least number of independent sets. In the same theorem, we will show that the tree with the most number of independent sets is $K_{1,n-1}$, the star on n vertices (a graph introduced in the bipartite section of the introduction). Among all graphs, as we will note in the same section, the graph with most number of independent sets is E_n and the graph with the least number of independent sets

is K_n . Thus, it is useful to end this section with some quick counts of the number of independent sets in these graphs.

Proposition 2.1.3. *For any positive integer n ,*

1. $i(E_n) = 2^n$,
2. $i(K_n) = n + 1$ and
3. $i(K_{1,n-1}) = 2^{n-1} + 1$.

Proof. We notice that every subset of E_n is independent as it contains no edges, and so $i(G)$ is exactly the number of subsets of an n -element set. As shown in Appendix C, the number of a set with n elements is 2^n , and so $i(E_n) = 2^n$. On the other hand, every two vertices of $V(K_n)$ are adjacent, and so the only independent sets of K_n are \emptyset and sets consisting of one vertex, of which there are n . Thus, $i(K_n) = n + 1$. Finally, consider $i(K_{1,n-1})$. Label the vertices of the partite set of size $n - 1$ as v_1, v_2, \dots, v_{n-1} , and the vertex in the other partite set v . As no vertices of $\{v_1, \dots, v_{n-1}\}$ are adjacent, this set forms an E_{n-1} inside the graph. By our previous argument, this yields 2^{n-1} independent sets. As v is adjacent to every vertex of the graph other than itself, it only participates in one independent set, that with no other vertices (of size one). Thus $i(K_{1,n-1}) = 2^{n-1} + 1$. \square

2.2 Bounding the quantity $i(G)$

The initial results we've looked at have all dealt with a direct count of $i(G)$ for certain types of graphs. While direct counts such as these are often aesthetically pleasing, and are often useful, there is another type of question that is much more common. Given a particular family of graphs, we often ask which graph in the family maximizes or minimizes $i(G)$. Among all graphs (the family of graphs with no restrictions), the question is not difficult to answer. An observation that is quickly made by any mathematician who studies independent sets is that adding edges to a graph G can only decrease $i(G)$ (and any $i_t(G)$ for that matter), for independent sets are, by definition, sets of non-adjacent vertices. Thus, as any n -vertex graph can be obtained from E_n by adding edges, and from K_n by deleting edges, $i(K_n) \leq i(G) \leq i(E_n)$. For reference, we present this observation here as a proposition.

Proposition 2.2.1. *For any graph G on n vertices,*

$$n + 1 = i(K_n) \leq i(G) \leq i(E_n) = 2^n.$$

The bounds in this proposition follow from Proposition 2.1.3. The interesting questions concerning which graphs in a family maximize or minimize $i(G)$ come from putting different restrictions on the on families of graphs we consider. The first class

of graphs to be well-studied were trees. The first major result on bounding $i(G)$ was the following, due to Prodinger and Tichy, in [2].

Theorem 2.2.2 (Prodinger and Tichy 1982). *For any tree T on n vertices,*

$$F_{n+1} = i(P_n) \leq i(T) \leq i(K_{1,n-1}) = 2^{n-1} + 1.$$

That $i(T)$ among trees is maximized by $K_{1,n-1}$ will turn out to be a corollary of our main original result (which can be found in Chapter 6.1) together with one of the most well known results in graph theory. We will, however, prove this and the other assertion, that P_n minimizes the number of independent sets among trees, with Prodinger and Tichy's original proof, as also found in [2], here. We ask the reader to keep in mind that this, and many other of the initial results of independent set theory will prove to be corollaries of more modern results. This will be a major theme of this work, and will be discussed in more detail later on.

Proof. We first prove that $i(T) \leq i(P_n)$ for any n -vertex tree and $n \in \mathbb{N}$. In light of Proposition 2.1.1, which states that $i(P_n) = F_{n+1}$, it is equivalent to show that for any n -vertex tree, $F_{n+1} \leq i(T)$. We do so by induction on n . We will actually show a slightly stronger result in this direction, by considering forests. The result holds trivially for forests of order $n \in \{1, 2\}$, as forests on these numbers of vertices are necessarily paths. Assume that the result holds for all forests on $n \leq k$ vertices for some $k \in \mathbb{N}$, and consider a forest T of order $k + 1$. The goal is to show that $i(T) \geq F_{k+2}$. We note that, as was shown in Proposition 1.1.3, either T is E_n , or there must be some vertex $v \in V(T)$ such that $d_T(v) = 1$. If $T = E_n$, then the result holds immediately from Proposition 2.2.1, and so we may assume that we have a v of degree one. Say that $w \in V(T)$ is the one vertex of $V(T)$ adjacent to v , and consider the subgraph obtained from T by removing v and the one edge adjacent to it, which we will denote $T - \{v\}$. Also, consider the subgraph obtained from $T - \{v\}$ by removing w and all adjacent edges, which we will denote similarly as $T - \{v, w\}$.

We bound $i(T)$ by bounding the number of independent sets which contain v , and the number which don't, much like we did in the proof of the first proposition of this section. As the only neighbor V is w , the number of independent sets which contain v is exactly $i(T - \{v, w\})$. The number which don't is, by definition, $i(T - \{v\})$. Trivially, $T - \{v\}$ and $T - \{v, w\}$ are both forests with orders n and $n - 1$, respectively, and thus, by our induction assumption,

$$i(T) = i(T - \{v\}) + i(T - \{v, w\}) \geq F_{k+1} + F_k = F_{k+2},$$

as desired. The proof of the lower bound on $i(T)$ is complete.

We now show the upper bound, that $i(T) \leq i(K_{1,n-1})$ for any n -vertex tree and $n \in \mathbb{N}$, again by induction, using similar techniques. It is equivalent to show that $i(T) \leq 2^{n-1} + 1$ as $i(K_{1,n-1}) = 2^{n-1} + 1$ by Proposition 2.1.3. As above, it holds easily for $n = 1$ as the only independent sets in a graph on one vertex are that vertex

and \emptyset . Assume that it holds for any tree of order less than or equal to k , and let T be a tree of order $k + 1$. The goal is to show that $i(T) \leq 2^k + 1$. As above, we can consider some vertex $v \in V(T)$ of degree one, its one neighbor w , and note that $i(T) = i(T - \{v\}) + i(T - \{v, w\})$, which we know exists by our Chapter 1.1 result, Theorem 1.1.3, that all trees contain vertices of degree one. As v is a vertex of degree one, $T - \{v\}$ is surely a tree (of order n , and so by our induction assumption, $i(T - \{v\}) \leq 2^{k-1} + 1$. Further, by Proposition 2.2.2, because $T - \{v, w\}$ is some $(n - 1)$ -vertex graph, $i(T - \{v, w\}) \leq 2^{k-1}$, and so, putting this all together,

$$i(T) = i(T - \{v\}) + i(T - \{v, w\}) \leq (2^{k-1} + 1) + 2^{k-1} = 2 \cdot 2^{k-1} + 1 = 2^k + 1,$$

as desired. \square

With a small amount of work, Prodinger and Tichy's theorem quickly extends to one which is significantly stronger, about connected graphs. This, together with the observation that K_n is a connected graph, allowing for a lower bound by Proposition 2.2.1, gives us the following.

Corollary 2.2.3. *If G is any n -vertex connected graph, then*

$$n + 1 = i(K_n) \leq i(G) \leq i(K_{1,n-1}) = 2^{n-1} + 1.$$

Proof. We consider any connected graph G . As was remarked in the explanation preceding Proposition 2.2.1, removing edges from a graph G can only increase $i(G)$, and thus, if we show that we can obtain a tree T by removing edges from a connected graph G , then we have shown that G has less independent sets than some tree, and thus less independent sets than $K_{1,n-1}$ by Theorem 2.2.2. It suffices to show that if G contains x number of cycles, for any positive integer x , then we can always decrease the number of cycles in G to $x - 1$ while still leaving G connected.

So, assume that connected graph G contains x cycles, for some positive integer x . Consider one of the cycles, call it C , on vertices $v_1, v_2, \dots, v_k \in V(G)$. Remove the edge $e := v_1 v_2$. We will denote this obtained graph by $G - e$. Clearly, $G - e$ has at most $x - 1$ cycles, as desired. We show that $G - e$ is still connected. Consider any $u, v \in V(G)$. There was a u, v -path say $P_{u,v}$ in G by definition of connectedness. If that $P_{u,v}$ did not traverse e , then $P_{u,v}$ is a path in $G - e$. If the $P_{u,v}$ did, say then at the point at which e was traversed, replace edge e with path $v_2, v_3, \dots, v_k, v_1$ defined by C . This is a u, v -walk in $G - e$, implying that $G - e$ is connected, and proving that $i(G) \leq i(K_{1,n-1})$. That $i(K_{1,n-1}) = 2^{n-1} + 1$ follows from Proposition 2.1.3. \square

Though it is not particularly relevant here, we notice that the proof of this corollary implies that a tree is the connected graph of minimal size, a result that is commonly used throughout graph theory. Also, before moving on to more specific restrictions, we would like to note that every bound presented thus far is what graph theorists call *sharp*, meaning that the bounds are actually obtainable by some graphs, and thus the bounds cannot be improved.

These initial results, and many more like them, as well as their numerous applications to real world problems, led to the continuing study of which graphs in many other classes of graphs maximize and minimize $i(G)$. More often, the question of which graphs maximize $i(G)$ is more interesting to the mathematical community for various reasons, and so this question is more studied than the minimization question (though, interestingly, the question of which graph minimizes is usually much harder). Naturally, the main related questions for forests were soon answered, as they are the generalization of trees; and even the question of which unicyclic (one cycle) graphs maximize the total number on independent sets has been answered, but this did not happen until 2006, in [4]. We state the result here without proof.

Theorem 2.2.4 (Pedersen and Vestergaard 2006). *If G is a graph which contains one cycle, and that cycle has length k , then*

$$i(G) \leq 2^{n-k} F_{k+1} + F_{k-1}.$$

This result is particularly interesting for two reasons. The first, is that it shows that the total number of independent sets in a graph depends on the length of its cycles in some way. It turns out that there are very strong relationships between the structures of the cycles in graphs and the structures of the independent sets, and this result provides a small hint into that. Another very strong insight comes from König's theorem which, as previously stated, will appear in a later section. The second reason that this theorem is interesting, is that it shows that the total number of independent sets of a graph does intrinsically have something to do with Fibonacci numbers, even when we add cycles. When many cycles are added, this is almost impossible to see, but we can still preserve the insight when one is added. It is a very thought-provoking point that is very difficult, if at all possible to pin down.

Other classes of graphs on which the question of which graph maximizes $i(G)$ has been studied include, but are not limited to, graphs with a fixed average degree, graphs with a given number of cut-edges, regular graphs, and graphs with given minimum degree. These later classes are the focus of Chapter 5.1. Many questions are still open in all of these classes, and, as in most areas of mathematics, there is no limit to the number of questions that can be asked. Which questions are being asked, and which questions will be asked, will inevitably depend on which prove to be interesting, from either a pure or applied perspective, based on work that is currently being done, and all the work that has been done since 1982.

Chapter 3

Independent sets of a fixed size

As stated in the previous section, the quantity $i(G)$ was studied before $i_t(G)$, and so, when waves of questions began to appear concerning the enumeration of independent sets in various classes of graphs, the questions were related to $i(G)$. This is because mathematics is a subject that builds upon itself, and so it is not surprising that substantially more work has gone into the study of $i(G)$ because it was the first to be considered. We expect that more and more work related directly to $i_t(G)$ will be done with time.

3.1 On the study of $i_t(G)$

Before looking general bounds for $i_t(G)$ for varying values of $t \in \mathbb{N}$, many of which are closely related to those presented above, let us first try to understand the sequence $(i_t(G))_{t \in \mathbb{N}}$ a little better. We make some initial observations about the sequence, which would usually be made by anyone who studies independent set theory, and package them as a proposition. For those readers who are new to independent set theory, the proofs of these will provide a nice warm-up for this section and those to come. One of the bounds in the following proposition uses a binomial coefficient, as do many bounds in statements and proofs of results throughout this paper. For an introduction to binomial coefficients, see Appendix C.

Proposition 3.1.1. *For any graph G on n vertices,*

1. $i_0(G) = 1$, $i_1(G) = n$ and $i_2(G) = \binom{n}{2} - e(G)$,
2. $i_t(G) = 0$ for any $t > \alpha(G)$ and
3. For any $t \in \mathbb{N}$, $0 \leq i_t(G) \leq i_t(E_n) = \binom{n}{t}$.

Proof. We show (1). As noted in the proof of Proposition 2.1.1, the empty set is an independent set of size zero (and is the only independent set of size zero), and

each vertex of any graph is an independent set of size one (and they are the only independent sets of size one), so $i_0(G) = 1$ and $i_1(G) = n$. To show that $i_2(G) = \binom{n}{2} - e(G)$, we note that independent sets of size two are exactly pairs of non-adjacent vertices. We can compute the number of non-adjacent vertices, by considering the number of all pairs of vertices in the graph, $\binom{n}{2}$, and subtracting away the number of adjacent pairs, which is exactly the number of edges (for two vertices are adjacent if and only if there is an edge between them). This proves (1).

We note that (2) follows directly from the definition of $\alpha(G)$. That is, as $\alpha(G)$ is the largest independent set in a graph, there cannot be any independent sets of larger size than $\alpha(G)$. To show (3), first note that an independent set cannot have negative size by definition, and so the only thing to show is $i_t(G) \leq i_t(E_n) = \binom{n}{t}$. As remarked in the proof of Proposition 2.2, adding edges to a graph can only decrease $i_t(G)$ for any t , and so $i_t(G) \leq i_t(E_n)$. Further, as any subset of E_n is independent, $i_t(E_n) = \binom{n}{t}$. \square

We can completely understand the starting values of our sequence $(i_t(G))_{t \in \mathbb{N}}$ by (1), and so that it is only natural, before continuing, that we would want to understand our ending value, $\alpha(G)$ a little better in general. Our next two sections are devoted to this.

3.2 Independent sets of maximum size

Proposition 3.1.1 tells us that when we are studying $(i_t(G))_{t \in \mathbb{N}}$, we actually only need to study $(i_t(G))_{t \in \{2, \dots, \alpha(G)\}}$. As a result, a natural question to ask is how large or small the independence number, $\alpha(G)$, can be under various restrictions. It turns out, that questions about $\alpha(G)$ were begin asked and answered well before $i(G)$ was even being considered. This can be seen, for example, by looking at the following theorem of Turán, one of the pioneers of graph theory, which he published over 40 years before any of the results in the previous section appeared. For any reader who is interested in studying random graphs, or who enjoys studying the probabilistic method in general, we note that there is a very charming probabilistic proof of this theorem, as presented in, for example, [5]. However, this theorem can actually be proven with terminology no more advanced than what has already been presented in this paper. We present such a proof here.

Theorem 3.2.1 (Turán 1941). *For any graph G ,*

$$\alpha(G) \geq \frac{n(G)}{\Delta(G) + 1}.$$

Proof. Let G be any graph, and let I be an independent set in G of maximal size. That is, let $|I| = \alpha(G)$. We note that any vertex of $V(G) \setminus I$ must be adjacent to a vertex of I , for if some $v \in V(G) \setminus I$ was not adjacent to any vertex of I , $I \cup \{v\}$ would be a larger independent set, contradicting the maximality of $|I|$. It follows, as I is independent, that $N_G(I) = V(G) \setminus I$, where $N_G(I) := \cup_{v \in I} N_G(v)$, and moreover, that $|I| + |N_G(I)| = n(G)$. As each vertex of I can be adjacent to at most $\Delta(G)$ vertices by definition, this gives us the bound $n(G) \leq \alpha(G) + \alpha(G)\Delta(G) = \alpha(G)(1 + \Delta(G))$, or, $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$. \square

As is the case with $i(G)$ and $i_t(G)$, we can explicitly determine the $\alpha(G)$, in terms of n , for different specified n -vertex graphs. For cycles and complete bipartite graphs in particular, an explicit value for the independence number will prove very useful. We look at the independent numbers of those graphs now.

Proposition 3.2.2. *For any positive integer n ,*

- $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$, and
- $\alpha(K_{a,b}) = \max\{a, b\}$ for any positive integers a and b .

Proof. Let n be any positive integer. Consider C_n on vertices v_1, v_2, \dots, v_n . Say that I is an independent set of maximum size in C_n . We know that there must be at least one vertex in I . That is, that for some $i \in \{1, 2, \dots, n\}$, $v_i \in I$. First assume that n is even. We must show that, in this case, that $\alpha(C_n) = \frac{n}{2}$. If i is odd, then, since I is of maximum size, it must include exactly all odd labeled vertices, and so in this case,

$I = \{v_1, v_3, \dots, v_{n-1}\}$, which has size $\frac{n}{2}$, as desired. A similar argument holds if i is even. We now assume that n is odd, and in this case, show that I has size $\frac{n-1}{2}$. If i is odd, then by similar reasoning used in the n is even case, $I = \{v_1, v_3, \dots, v_{n-2}\}$, which has size $\frac{n-1}{2}$. In this case, I cannot be even, for it would follow by the same reasoning that $I = \{v_2, v_4, \dots, v_{n-1}\}$ which has size $\frac{n-3}{2} < \frac{n-1}{2}$. This proves the assertion about C_n .

Consider $K_{a,b}$ for any positive integers a and b . Say that we have partite sets A and B , of sizes a and b , respectively. By definition, every vertex of A is adjacent to every vertex of B , and so the only independent sets of this graph lie entirely in A or entirely in B . However, since A and B are themselves independent sets (also, by definition), we know that the maximum independent set of the graph is one of these. It follows that $\alpha(K_{a,b}) = \max\{|A|, |B|\} = \max\{a, b\}$. \square

3.3 Independent sets of various sizes

Bounding $\alpha(G)$ for different families of graphs is an entire field of study in itself, and so we do not go any deeper into these questions surrounding $\alpha(G)$ then we have so far. Instead, we return our focus to some general bounding of $i_t(G)$ in classes of graphs where it is not terribly difficult. We start with a result which generalizes the upper bounds presented in Chapter 2.2. From this point, we do not deal with the lower bound analogues as they are much different questions, and as all of the original results we are building to deal only with questions of maximization.

Theorem 3.3.1. *For any n -vertex tree T and any $t \in \mathbb{N}$,*

$$i_t(G) \leq i_t(K_{1,n-1}).$$

It follows that the same bound holds for any connected graph G .

Remark 3.3.1. We note that, as $i(G) = \sum_t i_t(G)$, Theorem 2.2.2 and its corollary, which state the analogous results for $i(G)$, become a corollary of this.

That $i(T)$ among trees is maximized by $K_{1,n-1}$ can be viewed as a corollary of the main original result of this work, which can be found in Chapter 6.1, together with one of the most well known results in graph theory, which is most often referred to as König's theorem, a result we will discuss in Chapter 7.1. We will prove it as a corollary of these in that section. The reason that this becomes a corollary of our main result, is that our result provides a sharp bound for any bipartite graphs with given minimum degree, and it turns out that all trees are bipartite graphs which have minimum degree one (which we explain when we present this as a corollary). The fact that these classical results can be thought of as corollaries to our new theorem provides a perfect example of how current research in this field is not just providing new results, but is providing a bigger, clearer picture, in which older results are captured. These broader theorems get mathematicians closer to the seemingly unattainable, big pictures that overlay the subject.

It turns out that in the same way we can generalize these results, and place these graphs into a broader class of all similar graphs with the same minimum degree, it can be shown that these classic results can also be extended to a much broader, completely different classification of graphs by a 2011 theorem of Cutler and Radcliffe, proved in [6], which also leaves these upper bounds as corollaries. We first state this powerful theorem, and then we explain why it generalizes these bounds. In order to state this theorem, however, we need to first define a graph called the lex graph.

Definition. For any $n, m \in \mathbb{N}$, the *lex graph*, denoted $L(n, m)$, is defined to be the n -vertex graph on m edges obtained by letting $V(L(n, m)) = \{1, 2, \dots, n\}$ and by iteratively adding edges

$$12, 13, 14, \dots, 1n, 23, 24, \dots, 2n, 34, \dots,$$

until m edges have been assigned. For example, $L(3, 4) = K_{1,3}$.

Those familiar with the concept of a lexicographic ordering can easily see where this graph gets its name. We are now ready to state the theorem.

Theorem 3.3.2 (Cutler and Radcliffe 2011). *For any n -vertex graph G with m edges,*

$$i_t(G) \leq i_t(L(n, m))$$

for any $t \in \mathbb{N}$.

It is not obvious a priori why this result would make Theorem 2.2.2 (and thus all these other classical theorems by Remark 3.3.1) a corollary, but it turns out that we can show this without too much work. We do so now.

Proof of Theorem 2.2.2. Once we show that for any n -vertex tree T and any $t \in \mathbb{N}$ that $i_t(G) \leq i_t(K_{1,n-1})$, we have that the same bound holds for any connected graph G , as explained in the proof of Corollary 2.2.3. We do this by first showing that any n -vertex tree has the same number of edges, specifically $n - 1$, which implies by Theorem 3.3.2 that for any n -vertex tree T , $i_t(T) \leq i_t(L(n, n - 1))$; and then by showing that, by definition, $L(n, n - 1) = i_t(K_{1,n-1})$, as desired.

We show that all trees of $n - 1$ edges by induction. This is easily true for $n = 1$. Assume true for every tree of order n or less. Consider any $(n + 1)$ -vertex tree T . We know, as shown in the first section of this work, that T must contain some vertex v of degree one. As was the technique used in the proofs of Propositions 2.1.1 and 2.1.2, consider the n -vertex graph $T - \{v\}$ obtained by removing v and the one edge incident to it. This has $n - 2$ edges, implying that with the removed edge replaced, T has $n - 1$. This assertion is proven.

It remains to show that $L(n, n - 1) = K_{1,n-1}$. We consider a set of n vertices labeled by $\{1, 2, \dots, n\}$. We add edges $12, 13, 14, \dots, 1(n - 1)$, and have exactly enough to do this, but then have no more edges. We have clearly created $K_{1,n-1}$. \square

As one would think, Theorem 3.3.2 has many consequences beyond the ones mentioned, but the number of independent sets in graphs with a fixed number of edges is a deep, well-studied field in itself, and so we look no deeper into corollaries of this theorem.

Though there are some more results bounding $i_t(G)$, in most of the classes mentioned in Chapter 2.2 where strong bounds have been obtained for $i(G)$, no bounds have yet been obtained for $i_t(G)$, and thus there are many open questions. It is usually the case that $i_t(G)$ is more difficult to count, but strong bounds on $i_t(G)$ do provide strong bounds for $i(G)$ as we have seen, and continue to see in the following sections. The main result of this work is a result of this type, and has a fairly complex proof for this reason, but provides many nice corollaries which follow almost immediately. To end this section, we temporarily turn away from direct bounds for $i_t(G)$, and we turn our attention to one of the most well studied structures related to independent

sets of a fixed size, the independence polynomial, about which we will be able to use our current work to say different things about later.

Chapter 4

On the independence polynomial of a graph

As stated in the introduction, the polynomial defined by a graph G , which we can (in light of Proposition 3.1.1) now write with more definitive bounds as

$$P(G, x) := \sum_{t=0}^{\alpha(G)} i_t(G) x^t,$$

is referred to as the independence polynomial of G . It is part of a broad class of functions called *generating functions*, which are particularly important to combinatorialists. They are the formal power series whose coefficients encode information about a particular sequence of numbers, in this case, $(i_t(G))_{t \in \mathbb{N}}$. We will not go into generating functions in great detail, especially properties which do not relate directly to $P(G, x)$, but we will say that they are often most useful in finding closed forms or providing other, less rigid, structural information for the sequences they encode. For example, given the Fibonacci numbers, as defined in Chapter 2.1, we can use generating functions to show that

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}, \quad \text{where} \quad \phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1 - \sqrt{5}}{2}.$$

The interested reader can find more information about generating functions in general in, for example, [7], which is a well-written text that provides a friendly introduction to very basic generating function theory, as well as a detailed description and study of some of the more advanced topics of the field.

4.1 Important observations and a brief history

The specific generating function with which we are concerned, the independence polynomial, was first introduced by Gutman and Harary in 1983, in [8], not long after the study of $i_t(G)$ first began. We start our study of it with the following observations.

Proposition 4.1.1. *For any graph G and associated independence polynomial $P(G, x)$, the following properties hold.*

1. $P(G, 1) = i(G)$,
2. If $i_t(H) = i_{t-1}(G)$, then $P(H, x) = xP(G, x)$, for any positive integer t .

Proof. We note that, not just does this polynomial generate the independent sets of any fixed size, but for any graph G , letting $x = 1$ gives us exactly the total number of independent sets. That is, for any G ,

$$P(G, 1) = \sum_{t=0}^{\alpha(G)} i_t(G) = i(G).$$

This proves (1). To prove (2), we just notice that, by definition, $i_t(H) = i_{t-1}(G)$ implies that the coefficients of x^t in $P(H, x)$ is the coefficient of x^{t-1} in $P(G, x)$. \square

This proposition will prove quite useful in coming sections, and in the next theorem, in which we directly calculate the independence polynomial for paths. We saw in Chapter 2.1 that $i(P_n) = F_{n+1}$, the $(n+1)$ st Fibonacci number. It turns out that the independence polynomial of P_n (for any positive integer n), $P(P_n, x)$, is exactly, what is called, the $(n+1)$ st Fibonacci polynomial, denoted $F_{n+1}(x)$. The Fibonacci polynomials are defined by a recurrence relation, just as the Fibonacci sequence is, for all non-negative integers n , as follows:

$$F_n(x) := \begin{cases} 1 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-1}(x) + xF_{n-2}(x) & \text{if } n \geq 2. \end{cases}$$

For some explicit examples, let us look at the two small Fibonacci polynomials,

$$F_2(x) = x + 1 \quad \text{and} \quad F_3(x) = 2x + 1,$$

which we are claiming are $P(P_1, x)$ and $P(P_2, x)$, respectively. As $i_0(P_1) = i_0(P_2) = 1$, it is clear that we have the desired constant terms. In fact, it is not hard to see that the constant term of any Fibonacci polynomial will always be one, as it should be since any graph has exactly one independent set of size zero. As any P_n has exactly n independent sets of size one, we also see that the coefficients of the x terms are correct. There are no independent sets of size larger than one in either small path,

and so we have verified that our claim holds for these two small examples. We now show the result in general, which was proven by Arocha in a paper published two years after Prodinger and Tichy's proof that $i(P_n) = F_{n+1}$, [2].

Theorem 4.1.2. *For any positive integer n ,*

$$P(P_n, x) = F_{n+1}(x).$$

Proof. We show this by induction on n using techniques similar to those used in the initial $i(G)$ results discussed in Chapter 2.1. If $n = 1$, the result holds by the comments of the preceding paragraph. Assume true for positive integers less than n , and consider P_{n+1} on vertices v_1, \dots, v_{n+1} . We notice that

$$\begin{aligned} i_t(P_{n+1}) &= |\{I \in \mathcal{I}_t(G) : v_{n+1} \notin I\}| + |\{I \in \mathcal{I}_t(G) : v_{n+1} \in I\}| \\ &= i_t(P_n) + i_{t-1}(P_{n-1}). \end{aligned}$$

The second equality holds as any $I \in \{I \in \mathcal{I}_t(G) : v_{n+1} \notin I\}$ contains exactly t non-adjacent vertices of $P_n - \{v_{n+1}\}$, and any $I \in \{I \in \mathcal{I}_t(G) : v_{n+1} \in I\}$ contains exactly $t-1$ non-adjacent vertices of $P_{n+1} - \{v_n, v_{n+1}\}$. By Proposition 4.1.1 (2), this implies that

$$\begin{aligned} P(P_{n+1}, x) &= P(P_n, x) + xP(P_{n-1}, x) \\ &= F_{n+1}(x) + xF_n(x) \\ &= F_{n+2}(x), \end{aligned}$$

with the second equality holding by our inductive assumption, and the last holding by definition of the Fibonacci polynomial. \square

Remark 4.1.1. Using this result, together with some rather advanced generating function techniques, one can show that for any non-negative integer n ,

$$P(P_n, x) = \sum_{t=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-t}{t} \cdot x^t.$$

See **CITE IT** for details. This implies that for any positive integer n ,

$$i_t(P_n) = \binom{n+1-t}{t},$$

which is a very convenient result to have, for obvious reasons.

It turns out that, in the same way that $i(C_n)$ can be defined in terms of a recurrence relation that is based on the Fibonacci numbers (specifically, the Lucas numbers, as shown in Chapter 2.1), the independence polynomial of C_n can be defined in terms of a recurrence relation based on the Fibonacci polynomials. Specifically,

$$P(C_n, x) = F_{n-1}(x) + 2xF_{n-2}(x).$$

The proof of this is very similar to the proof of Theorem 4.1.2, and so we will omit it. To see the proof precisely, see [9].

It turns out that finding a closed form for the independent set sequence of a given graph, like the ones we obtained for F_n and $F_n(x)$ using generating functions, is not something that is practical in most cases. For this reason, other structural questions about $(i_t(G))_{t \in \mathbb{N}}$ are asked. The independence polynomial will often help us study as well. Some of the most common types of questions about the sequence $(i_t(G))_{t \in \mathbb{N}}$ have to do with what is called unimodality. We say that a sequence $(a_t)_{t \in \mathbb{N}}$ is *unimodal* if there is some $t_0 \in \mathbb{N}$ such that, either

$$a_0 \leq a_1 \leq \cdots \leq a_{t_0-1} \leq a_{t_0} \quad \text{or} \quad a_0 \geq a_1 \geq \cdots \geq a_{t_0-1} \geq a_{t_0}$$

and, either

$$a_{t_0} \leq a_{t_0+1} \leq \cdots \leq a_{\alpha-1} \leq a_{\alpha} \quad \text{or} \quad a_{t_0} \geq a_{t_0+1} \geq \cdots \geq a_{\alpha-1} \geq a_{\alpha},$$

Informally, $(a_t)_{t \in \mathbb{N}}$ if there is some $t_0 \in \mathbb{N}$ such that the sequence is monotonic on either side of the term a_{t_0} .

For an example of a graph with a unimodal independent set sequence, consider the complete bipartite graph $K_{l,l}$ with equal partite set sizes for any positive even integer l . To see why $(i_t(K_{l,l}))_{t \in \mathbb{N}}$. To see this, first recall that $i_t(K_{l,l}) = 2\binom{l}{t}$, as any subset of a partite set is independent (but no other subset is), and thus note that we can show $(i_t(K_{l,l}))_{t \in \mathbb{N}}$ is unimodal, as sequences of binomial coefficients indexed by increasing sets always define unimodal sequences (see Proposition C.3.1 in Appendix C). We can also use the unimodality of binomial coefficients to show the following

Proposition 4.1.3. *For any non-negative integer n , the associated independent set sequence $(i_t(P_n))_{t \in \mathbb{N}}$ is unimodal.*

Proof. As stated in Remark 4.1.1, it can be shown (with the right tools), that

$$i_t(P_n) = \binom{n+1-t}{t}.$$

Thus, the unimodality of $(i_t(P_n))_{t \in \mathbb{N}}$ follows immediately by arguments similar to those presented in the paragraph preceding this proof. \square

Many other classes of graphs, some of which are bipartite graphs, and many of which are not, are unimodal. For some examples, see [10]. We will present some original results on the independence polynomial that deal with bounding it in graphs with certain degree restrictions, and for certain values of x in Chapter 5.1, but we will present some results which are not directly related to degree restriction in Chapter 4.2. Particularly, in order to provide a brief introduction to the types of questions that are commonly asked about $P(G, x)$ (such as those about unimodality), and in order to introduce some other interesting concepts, open questions, and generate interest

in some topics which are not directly related, we provide a section about an open question, commonly called the Roller-Coaster Conjecture.

4.2 The Roller-Coaster Conjecture

Before we can talk about the Roller-Coaster Conjecture, we must first define two important terms. The first has to do with the structure of the independent sets in graphs.

Definition. We say that a graph G is *well-covered* if every independent set in G which is maximal is also maximum. Equivalently, every independent set I which has a neighborhood equal to $(V(G) \setminus I)$ has size $\alpha(G)$.

For some examples of well-coveredness, notice that K_n and E_n are both well-covered graphs for any positive integer n , and that if G is a complete bipartite graph with bipartition $V(G) = A \cup B$, then G is well-covered if and only if $|A| = |B|$ (because G has exactly two independent sets, A and B , which are both maximal). The other definition we need to consider is directly related to the sequence defined by the numbers of independent sets of fixed sizes.

Definition. Given a collection of graphs with the same independence number α , we say that, for some positive integer l , the indexing set i_1, i_2, \dots, i_l is *any-ordered* on the collection if for any permutation $\pi : \{1, \dots, \alpha\} \rightarrow \{1, \dots, \alpha\}$, there is a graph G in the collection with independent set sequence $(i_t(G))_{t \in \{1, \dots, \alpha\}}$ such that

$$i_{\pi(1)}(G) < i_{\pi(2)}(G) < \dots < i_{\pi(\alpha)}(G).$$

It was conjectured in [11] that well-covered graphs have unimodal independent set sequences. In [12] however, counterexamples were produced, and the following conjecture, which is now quite well-known, was made.

Conjecture 4.2.1 (The Roller-Coaster Conjecture).

- For any well-covered graph G , the terms of the associated independence sequence strictly increase from $i_0(G)$ to $i_{\lceil \frac{\alpha(G)}{2} \rceil}(G)$, and
- the independence sequence for well-covered graphs with independence number $\alpha(G)$ is any-ordered on $\{\lceil \frac{\alpha(G)}{2} \rceil, \dots, \alpha(G)\}$.

Michael and Traves, in [12], proved the first assertion of Conjecture 4.2.1 (about the independence sequence increasing), and proved the second assertion for $\alpha(G) \leq 7$. In [10], Matchett improved this bound, showing the result for well covered graphs G with $\alpha(G) \leq 11$.

We note, before concluding this section, that even though not all well-covered graphs have unimodal independent set sequences, there are subclasses of well-covered graphs which do. For example, all well-covered trees are unimodal. For a proof of this, see [13].

One very interesting result related to this topic is a result of [11], which was the primary motivation for the original conjecture. It is the following.

Theorem 4.2.1 (Alvai, Malde, Erdős 1987). *Given any positive integer α , the independent set sequence for graphs of independence number α is any-ordered on the index set $\{1, 2, \dots, \alpha\}$.*

This Theorem tells us that if you have a graph G with given independent set sequence (starting from $i_1(G)$ rather than $i_0(G)$), then, no matter how you permute this independent set sequence, there is another graph of independence number $\alpha(G)$ which has that permutation as its independent set sequence. This is a very surprising result. One particularly nice proof of this result can be found in [10].

Chapter 5

The imposition of degree restrictions

We now study the maximization of the quantities focused on in the previous sections in classes of graphs satisfying different degree restrictions. Imposing a maximum degree restriction alone provides almost no interesting information. It is clear that E_n satisfies any possible maximum degree requirement, giving us that, for any n -vertex graph of maximum degree at most Δ ,

$$i_t(G) \leq \binom{n}{t}, \quad i(G) \leq 2^n \quad \text{and} \quad \alpha(G) \leq n,$$

with all these bounds sharp in this broad class. The interesting questions come from restricting our study to different classes, and specifically, to imposing minimum degree restrictions; the latter being the main focus of our original results. The first subsection here, which deals with the history of this degree restricted study, will be rather conversational, as the proofs use very different methods than we cover in this work. The subsection that follows it will also be rather conversational, but will involve slightly more mathematics than the first. It deals with the conjecture that motivated the bulk of our original research.

5.1 The history of degree restricted study

The first class of degree-restricted graphs on which independent enumeration was studied was the class of regular graphs, and within the class of regular graphs, Kahn made the first big breakthrough when he proved the following result in [14].

Theorem 5.1.1 (Kahn 2001). *If G is an r -regular bipartite graph on n vertices with $r \geq 1$, then*

$$i(G) \leq i(K_{r,r})^{n/2r} = (2^{r+1} - 1)^{n/2r}.$$

This bound is sharp, as it is achieved by taking disjoint copies of $K_{n,n}$ when r is divisible by $2n$. In fact, Kahn proved a more general result for weighted independent sets. His proof was one that used advanced probabilistic techniques, particular, some very clever, very original applications of what is called entropy. Nine years after this result was published, Zhao extended it from bipartite graphs to all graphs, using what is called the *bipartite double-cover*, in [15]. Given a graph G , the bipartite double cover of G , denoted $G \times K_2$, is the graph with vertex set $V(G) \times \{0, 1\}$ with $(u, i) \sim (v, j)$ if and only if $uv \in E(G)$ and $i \neq j$. The key lemma of Zhao used to prove Theorem 5.1.1 for arbitrary G was the following. It is here where the bipartite double cover is directly applied.

Lemma 5.1.2 (Zhao 2010). *If G is any graph, then*

$$i(G)^2 \leq i(G \times K_2),$$

with equality if and only if G is bipartite.

There were multiple papers on this subject that appeared in between 2001 and 2010 that dealt with similar topics. One year after Zhao's extension of Kahn's result was published, in [16], Galvin presented some results on enumerating independent sets in graphs with given minimum degree. This paved the way for research, some of which was ours, that would center around the relaxation of the maximum degree requirement, thus leaving the strong condition of regularity. His main result on this topic will be covered in the next section, along with a corollary of it and a related conjecture. These results provided a great deal of motivation for our original results to come.

5.2 On a 2011 conjecture

We consider the following conjecture, made by Galvin in [16]. This conjecture asserts that among all n -vertex graphs of minimum degree at least δ , the one with the most independent sets is $K_{\delta, n-\delta}$, provided that $n \geq 2\delta$.

Conjecture 5.2.1 (Galvin 2011). *If G is a graph on n vertices with minimum degree at least δ , where n and δ satisfy $n \geq 2\delta$, then*

$$i(G) \leq i(K_{\delta, n-\delta}).$$

Remark 5.2.1. The reason that we must require that $n \geq 2\delta$ is that if $n < 2\delta$, then $K_{\delta, n-\delta}$ has a partite with less than δ vertices, (namely, the $(n - \delta)$ -size partite set). In this case, the vertices in the other partite set would be adjacent to less than δ vertices, implying that $K_{\delta, n-\delta}$ would not be a graph of minimum degree at least δ (and thus would not be in our class of graphs). So, this requirement that $n \geq 2\delta$ is not so much a restriction, but more a technical point to guarantee that $K_{\delta, n-\delta}$ is a graph of minimum degree at least δ .

This was the conjecture that sparked us to begin research that led to all original results to come. Let us get some intuitive understanding of this conjecture, and explain why we believed, from the day we first stumbled upon it, that it was true. This theorem asserts, informally, that the way to have an n -vertex graph that maximizes $i(G)$ under the degree restriction that $n \geq 2\delta$, is to create the biggest independent set possible in your graph, that is, maximize $\alpha(G)$. Equivalently, the idea is to create the biggest empty graph within our graph, so that we can choose many independent sets from it.

While this does seem like a good idea, a priori, there is no reason to believe that it is necessarily the best. That is, that maximizing $\alpha(G)$ should necessarily maximize $i(G)$. In fact, minimizing the total number of edges (and thus the number of overall adjacencies) seems just as logical, and these two are not equivalent ideas. For example, if we fix $\delta = 2$ and any $n \geq 4$, the the graph which minimizes the number of edges is is C_n (as every vertex has the minimum degree, two), while the graph which maximizes $\alpha(G)$ is, as stated, $K_{\delta, n-\delta}$. Specifically, $e(C_n) = n$ while $e(K_{\delta, n-\delta}) = \delta(n - \delta)$ by Proposition 1.1.5, and it is easy to check that $n - \delta > n$ for almost all values of n , as $n \geq 2\delta$; while $\alpha(C_n) \leq \frac{n}{2}$ and $\alpha(K_{\delta, n-\delta}) = n - \delta$ by Proposition 3.2.2, implying similarly that $\alpha(K_{\delta, n-\delta}) > \alpha(C_n)$ for any values of $n \geq 2$.

Thus, we would not be justified in believing that whatever graph maximizes independent number necessarily maximizes total number of independent sets without other evidence. We provide three strong pieces of evidence. Recall that in Chapter 2.1, we stated that one of the first results to ever be published on the bounding of total number of independent sets came from Prodinger and Tichy, who showed that among connected graphs, $K_{1, n-1}$ was the one which maximized the quantity. It turns out that this theorem extends to graphs which are not necessarily connected, but which have no isolated vertices (components with one single vertex), as shown in [4].

Theorem 5.2.1 (Pedersen and Vestergaard 2006). *If G is a graph with no isolated vertices, then*

$$i(G) \leq i(K_{1,n-1}).$$

This proof was very computational, and so we omit it here, but instead make a key observation about this theorem. If a graph has no isolated vertices, then it surely has minimum degree at least one, so we can restate the theorem as,

For any n -vertex graph G with minimum degree at least 1, $i(G) \leq i(K_{1,n-1})$.

This is exactly our desired conjecture when $\delta = 1$. The next piece of evidence comes from Hua, in [17].

Theorem 5.2.2 (Hua 2009). *If G is a connected graph such that the removal of any two edges cannot disconnect G , then $i(G) \leq i(K_{2,n-2})$.*

The evidence provided by this theorem is not quite as strong as the evidence we gain from the other two theorems we are presenting here, but it is definitely significant. Having minimum degree at least two is a slightly weaker condition than the one given, since being 2-edge-connected implies minimum degree at least two, so this is not quite the $\delta = 2$ version of the conjecture we would want it to be, but it is incredibly similar. The final piece of evidence, which is by far the strongest, comes from the same paper in which Galvin made the conjecture. It was the following.

Theorem 5.2.3 (Galvin 2011). *For any $\delta > 0$, there is a $n(\delta)$ such that, if G is a graph with $n(G) \geq n(\delta)$ and $\delta(G) \geq \delta$, then*

$$i(G) \leq i(K_{\delta, n(G)-\delta})$$

with equality holds if and only if $G = K_{\delta, n(G)-\delta}$.

This theorem states that the conjecture holds for sufficiently large n , this, of course, being quite ambiguous. Specifically, the the conjecture holds, i.e., every n -vertex graph G of minimum degree at least δ has a smaller number of independent sets provided that

$$n \geq (C - 1)\delta^2 + [(1 - D)C + 1 + D]\delta - D, \text{ where } C = \frac{\ln(2)}{\ln(2) - \frac{1}{2}} \text{ and } D = \frac{2\ln(\frac{1}{2})}{\ln(2)}.$$

This specific equation is not important, which is why we did not state it specifically at first. For this reason, and because it uses techniques very different from the ones we will use in coming sections, we will not present the proof here. What we should take from this is simply that the conjecture is true for all n larger than some certain positive integer. This is very strong evidence that the conjecture holds in general, for this shows that it is true for most values of n (specifically, all but finitely many cases).

5.3 A stronger conjecture

For the reason we have just discussed in 5.2, we had become very convinced that Galvin's original conjecture, Conjecture 5.2.1, was true. Through personal communication with Galvin, an even stronger conjecture was developed. We believed that this result may hold, not just for $i(G)$, but for $i_t(G)$, for every t .

Conjecture 5.3.1 (Galvin). *If G is an n -vertex graph of minimum degree at least δ , then for any positive integer t ,*

$$i_t(G) \leq i_t(K_{\delta, n-\delta})$$

with equality holding if and only if $G = K_{\delta, n-\delta}$ for $t \geq 3$.

We quickly realized, as did Engbers independently, that this conjecture, Conjecture 5.3.1, was not true. As discussed in paragraphs that immediately followed Remark 5.2.1, we can almost always find graphs which have less edges than $K_{\delta, n-\delta}$. Recall that, as independent sets of size two are simply non-adjacencies, for any graph G , $i_2(G) = \binom{n}{2} - e(G)$ (as discussed in Proposition 3.1.1). It follows that $K_{\delta, n-\delta}$ will not always maximize $i_2(G)$. For a small example, consider $n = 6$ and $\delta = 2$. Using this, we see that $i_2(K_{2,4}) = 7$, while $i_2(C_6) = 9$, while $\delta(C_6) \geq 2$. The smallest possible counterexample is depicted in Figure 5.1

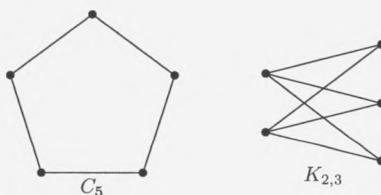


Figure 5.1: The cycle C_5 and the complete bipartite graph $K_{2,3}$

After some researching however, we did start to believe that the conjecture would hold for $t \geq 3$, as surprising as this would be. If this was the case, as i_0 and i_1 are the same for any n -vertex graph (as discussed in Proposition 3.1.1), this would mean that $K_{\delta, n-\delta}$ was the unique maximize for every non-negative integer t except 2. As strange as it seemed, the work we were doing was leading us to believe that this was true. The work consisted of many scattered ideas, small results, and pictorial observations in the early stages, and so we will not state them here. Instead, we will state the reformed version of Conjecture 5.2.1, and the following section will provide more evidence that it is true than any of those scattered ideas, results and observations possibly could.

Conjecture 5.3.2. *If G is an n -vertex graph of minimum degree at least δ , then for any $t \neq 2$,*

$$i_t(G) \leq i_t(K_{\delta, n-\delta})$$

with equality holding if and only if $G = K_{\delta, n-\delta}$ for $t \geq 3$.

We notice that none of the aforementioned results about independent set enumeration in degree restricted classes of graphs have dealt with $i_t(G)$. In fact, no main results from these papers, or any of the papers we have alluded to had any results of this type. This “level set version” of the conjecture, as we’ve come to call it, would be one of a very new breed of results in this area.

Chapter 6

Original results

As discussed at the beginning of the previous section, the first class of degree-restricted graphs on which independent enumeration was studied was the class of regular graphs, and within the class of regular graphs, Kahn made the first big breakthrough when he proved a result about bipartite graphs. We decided that we would approach the level set study the same way, and attempt to prove this the strong conjecture that we ended the section with, Conjecture 5.3.2 for bipartite graphs. We did so successfully. We begin this section with its statement and proof.

6.1 Our main original result

Theorem 6.1.1 (Alexander, Cutler and Mink). *Let n , δ , and t be positive integers with $n \geq 2\delta$ and $t \geq 3$. If G is a bipartite graph on n vertices and minimum degree at least δ , then*

$$i_t(G) \leq i_t(K_{\delta, n-\delta}),$$

with equality if and only if $G = K_{\delta, n-\delta}$.

Before we can offer a proof of this result, we need to first prove two very well-known binomial identities. The first is most commonly referred to as Vandermonde's identity, and it has a very intuitive explanation that will come out in the combinatorial proof we provide. The second, which we present here as a proposition, follows immediately from the factorial formula for binomial coefficients, Claim C.1.2 in the appendix. However, we provide a combinatorial proof of the identity so that we have an intuitive understanding of it when it is used in the proof of Theorem 6.1.1. The type of proof we use also provides a good introduction to counting techniques we will use in coming proofs.

Vandermonde's Identity. For any non-negative integers n and k ,

$$\binom{m+n}{k} = \sum_{j=0}^k \binom{m}{j} \binom{n}{k-j}.$$

Proof of Vandermonde's Identity. As explained in Appendix C, $\binom{m+n}{k}$ counts the number of subsets of size k of an $(m+n)$ -element set. We prove this identity combinatorially, by showing that $\sum_{j=0}^k \binom{m}{j} \binom{n}{k-j}$. This is not too difficult to see, as we can think of any set of $m+n$ elements as the union set of a set of n elements and a set of m elements. When choosing any k elements from the larger set, we choose some j of them from the set of m elements for $j \in \{1, 2, \dots, k\}$, and the other $k-j$ of them from the set of n elements. Summing over all possibilities, we see that the identity holds. \square

Proposition 6.1.2. For any non-negative integers n and k ,

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.$$

Proof. We show, specifically, that

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

Consider the n -element set $[n] := \{1, 2, \dots, n\}$. The quantity $k \binom{n}{k}$ exactly counts the number of ways to create a k -element subset of $[n]$ such that one element of the set is marked, or flagged, in some way. That is, $k \binom{n}{k}$ counts exactly the number of pairs (K, x) such that $K \subseteq [n]$, $|K| = k$, and $x \in K$. This is because $\binom{n}{k}$ is, by definition, the number of subsets K of $[n]$ such that $|K| = k$, and there are k ways to choose one element of K to mark. $n \binom{n-1}{k-1}$ counts the same set of pairs of the form (K, x) , for there are n ways to first choose the marked element x (before choosing the other $k-1$ elements of your set), and then there are $\binom{n-1}{k-1}$ ways to choose the other $k-1$ elements from $[n] \setminus \{x\}$. \square

Now that we have these tools ready, we can begin the proof the proof of our main result.

Proof of Theorem 6.1.1. Let G be any n -vertex bipartite graph of minimum degree at least δ and fix some positive integer $t \geq 3$. We show that $i_t(G) \leq i_t(K_{\delta, n-\delta})$. Say G has bipartition $G = A \cup B$. We may assume, without loss of generality, that $|A| \leq |B|$. We know that $|A| \geq \delta$, for if not, the vertices of B could not satisfy the minimum degree requirement, as discussed in Remark 5.2.1. Thus, for some integer c ,

$$|A| = \delta + c, \quad |B| = n - \delta - c \quad \text{and} \quad 0 \leq c \leq \frac{n - 2\delta}{2},$$

with the upper bound for c holding by the assumption $|A| \leq |B|$, which implies that $\delta + c = |A| \leq |B| = n - \delta - c$, or equivalently that $c \leq \frac{n-2\delta}{2}$, and the lower bound for c holding by our $n \geq 2\delta$ condition. Before continuing the proof, we make the following remark, which we will reference multiple times in this, and coming sections.

Remark 6.1.1. We notice that if $c = 0$, then the partite sets of G have sizes $|A| = \delta$ and $|B| = n - \delta$ by definition of c . Since all vertices of B must have minimum degree at least δ by assumption, and cannot be adjacent to any other vertices of B by definition of a partite set, it must be the case that every vertex of B is adjacent to every one of the δ vertices of A . That is, that $G = K_{\delta, n-\delta}$. On the other hand, if $c > 0$, then $|A| \neq \delta$ and $|B| \neq n - \delta$. Thus, $c = 0$ if and only if $G = K_{\delta, n-\delta}$.

We know that independent sets in G can be partitioned into those contained entirely in A , those contained entirely in B , and those containing vertices from both A and B . Let $\Lambda_t := \{I \in \mathcal{I}_t(G) : I \cap A, I \cap B \neq \emptyset\}$ (the independent sets of G which contain vertices of both A and B). As A and B are themselves independent sets, implying that any subset of A or subset of B is independent, we have that

$$i_t(G) = \binom{|A|}{t} + \binom{|B|}{t} + |\Lambda_t| = \binom{\delta + c}{t} + \binom{n - \delta - c}{t} + |\Lambda_t|.$$

Our goal is to show that

$$\binom{\delta + c}{t} + \binom{n - \delta - c}{t} + |\Lambda_t| \leq \binom{\delta}{t} + \binom{n - \delta}{t}.$$

So, we must bound $|\Lambda_t|$. To this end, we notice that we can define Λ_t as the independent sets of $\mathcal{I}(G)$ which contain exactly j vertices of the partite set B for $j \in \{1, \dots, t-1\}$. This is because any independent set of size t which contains exactly j vertices of B for $j \in \{1, 2, \dots, t-1\}$ contain exactly $t-j$ vertices of A for $(t-j) \in \{t-1, t-2, \dots, 1\}$. Letting j range over all these values surely gives all possible ways to have an independent set I for which the sets $I \cap A$ and $I \cap B$ are nonempty. Formally,

$$|\Lambda_t| = \sum_{j=1}^{t-1} |\{I \in \mathcal{I}_t(G) : |I \cap B| = j\}|. \quad (6.1)$$

We can represent this in a more convenient way, by noticing that

$$|\{I \in \mathcal{I}_t(G) : |I \cap B| = j\}| = \frac{1}{j} \sum_{b \in B} |\{(b, I) : I \in \mathcal{I}_t(G), |I \cap B| = j, b \in I\}|.$$

This equality holds because we can index each independent set which intersects B in exactly j elements by those j elements, but this results in counting that independent set j times, which is why we must divide by a factor of j . That is, if we index each

independent set by its vertices that lie in the partite set B , and sum over all $b \in B$, then we are exactly counting $j \cdot |\{I \in \mathcal{I}_t(G) : |I \cap B| = j\}|$. Substituting this into equation (1), we see that

$$|\Lambda_t| = \sum_{j=1}^{t-1} \frac{1}{j} \sum_{b \in B} |\{(b, I) : I \in \mathcal{I}_t(G), |I \cap B| = j, b \in I\}|.$$

Consider any $I \in \{(b, I) : I \in \mathcal{I}_t(G), |I \cap B| = j, b \in I\}$. The vertices of $I \cap A$ cannot be in the neighborhood of b and so, since $d_G(b) \geq \delta$, there are at most $|A| - \delta = \delta + c - \delta = c$ vertices in A from which to choose the $t - j$ vertices of $I \cap A$. We know that there are exactly $t - j$ vertices of I which must come from A as $|I| = t$ and $|I \cap B| = j$. Further, the vertices of $(I \cap B) \setminus \{b\}$ must not be in any of the neighborhoods of the vertices in $I \cap A$. This joint neighborhood must have size at least δ and so there are at most $n - 2\delta - c - 1$ vertices left to choose the $j - 1$ vertices of $(I \cap B) \setminus \{b\}$. Thus, by these two similar arguments, we have that

$$|\{(b, I) : I \in \mathcal{I}_t(G), |I \cap B| = j, b \in I\}| \leq \binom{c}{t-j} \binom{n-2\delta-c-1}{j-1} \text{ for } j \in \{1, \dots, t-1\}.$$

This implies that

$$|\Lambda_t| \leq \sum_{j=1}^{t-1} \frac{1}{j} \sum_{b \in B} \binom{c}{t-j} \binom{n-2\delta-c-1}{j-1}. \quad (6.2)$$

We notice that the summand $\binom{c}{t-j} \binom{n-2\delta-c-1}{j-1}$ does not depend on vertices of set B any longer, and so, recalling that $|B| = n - \delta - c$, we immediately have that

$$\sum_{b \in B} \binom{c}{t-j} \binom{n-2\delta-c-1}{j-1} = (n - \delta - c) \binom{c}{t-j} \binom{n-2\delta-c-1}{j-1}.$$

Putting this back into equation (2), we have that

$$|\Lambda_t| \leq \sum_{j=1}^{t-1} \frac{n - \delta - c}{j} \binom{c}{t-j} \binom{n-2\delta-c-1}{j-1}. \quad (6.3)$$

Using Proposition 6.1.2, we have that

$$\binom{n-2\delta-c}{j} = \frac{n-2\delta-c}{j} \binom{n-2\delta-c-1}{j-1},$$

or equivalently, that

$$\binom{n-2\delta-c-1}{j-1} = \frac{j}{n-2\delta-c} \binom{n-2\delta-c}{j}.$$

Substituting this directly into equation (3), we have that that

$$\begin{aligned}
|\Lambda_t| &\leq \sum_{j=1}^{t-1} \frac{n-\delta-c}{j} \binom{c}{t-j} \frac{j}{n-2\delta-c} \binom{n-2\delta-c}{j} \\
&= \frac{n-\delta-c}{n-2\delta-c} \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{n-2\delta-c}{j}
\end{aligned} \tag{6.4}$$

We will now use this bound on Λ_t to show that the difference $i_t(K_{\delta,n-\delta}) - i_t(G)$ is nonnegative, that is, that $i_t(K_{\delta,n-\delta}) \geq i_t(G)$, as desired. Recalling that $i_t(K_{\delta,n-\delta}) = \binom{\delta}{t} + \binom{n-\delta}{t}$, we have that

$$i_t(K_{\delta,n-\delta}) - i_t(G) = \binom{\delta}{t} + \binom{n-\delta}{t} - \binom{\delta+c}{t} - \binom{n-\delta-c}{t} - |\Lambda_t|$$

This, together with our bound on $|\Lambda_t|$ gives us that

$$\begin{aligned}
i_t(K_{\delta,n-\delta}) - i_t(G) &\geq \binom{n-\delta}{t} + \binom{\delta}{t} - \binom{n-\delta-c}{t} - \binom{\delta+c}{t} \\
&\quad - \frac{n-\delta-c}{n-2\delta-c} \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{n-2\delta-c}{j}
\end{aligned}$$

By applying Vandermonde's identity to the first and fourth term of this expression, we exactly see that this lower bound is equal to

$$\begin{aligned}
&\sum_{j=0}^t \binom{c}{t-j} \binom{n-\delta-c}{j} + \binom{\delta}{t} - \binom{n-\delta-c}{t} - \sum_{j=0}^t \binom{c}{t-j} \binom{\delta}{j} \\
&\quad - \frac{n-\delta-c}{n-2\delta-c} \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{n-2\delta-c}{j}.
\end{aligned}$$

We note that, as $t \geq 3$, our first summation does contain at least four terms. By extracting the first and final term (that is, the $j = 0$ and $j = t$ terms), we see that this first summation can be rewritten,

$$\sum_{j=0}^t \binom{c}{t-j} \binom{n-\delta-c}{j} = \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{n-\delta-c}{j} + \binom{c}{t} + \binom{n-\delta-c}{t}.$$

Applying an identical argument to the next summation of the bound gives

$$\sum_{j=0}^t \binom{c}{t-j} \binom{\delta}{j} = \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{\delta}{j} + \binom{c}{t} + \binom{\delta}{t}.$$

Substituting these equalities back into our bound, we see that many terms cancel. Our lower bound condenses to

$$\sum_{j=1}^{t-1} \binom{c}{t-j} \binom{n-\delta-c}{j} - \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{\delta}{j} - \frac{n-\delta-c}{n-2\delta-c} \sum_{j=1}^{t-1} \binom{c}{t-j} \binom{n-2\delta-c}{j}.$$

We see that the sum is common to all terms, as is $\binom{c}{t-j}$. We further condense all of these terms into one summand and factor out this binomial which is common to all of them. We obtain the much more convenient lower bound,

$$i_t(K_{\delta,n-\delta}) - i_t(G) \geq \sum_{j=1}^{t-1} \binom{c}{t-j} \left[\binom{n-\delta-c}{j} - \binom{\delta}{j} - \frac{n-\delta-c}{n-2\delta-c} \binom{n-2\delta-c}{j} \right].$$

Since $t \geq 3$, we know that $t-1 \geq 2$, and thus that there are at least two terms in this sum. Thus, we can separate (at least) the first two terms from the sum in the above expression, and obtain that

$$\begin{aligned} i_t(K_{\delta,n-\delta}) - i_t(G) \geq & -\binom{c}{t-1} \delta + \binom{c}{t-2} \left[\delta(n-2\delta-c) - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{2} \right] \\ & + \sum_{j=3}^{t-1} \binom{c}{t-j} \left[\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j} \right], \end{aligned} \quad (6.5)$$

if we note that

$$\sum_{j=1}^2 \binom{c}{t-j} \left[\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j} \right]$$

is exactly

$$-\binom{c}{t-1} \delta + \binom{c}{t-2} \left[\delta(n-2\delta-c) - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{2} \right].$$

We notice that this lower bound vanishes when $c = 0$, as we would expect, for we showed in Remark 6.1.1 that $c = 0$ if and only if $G = K_{\delta,n-\delta}$. We now assume that $c > 0$ for the remainder of this proof, and complete our argument by first showing that

$$\sum_{j=3}^{t-1} \binom{c}{t-j} \left[\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j} \right],$$

which we note has at least one term since $c > 0$, is positive; and then by showing that

$$-\binom{c}{t-1} \delta + \binom{c}{t-2} \left[\delta(n-2\delta-c) - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{2} \right] \geq 0.$$

This will imply that our lower bound as a whole is definitely positive, giving us that $K_{\delta, n-\delta}$ has strictly more independent sets of size t than any given $G \neq K_{\delta, n-\delta}$ in our class. The first of these inequalities will hold easily, once we prove the following claim.

Claim 6.1.1. If n , δ and j are positive integers such that $n \geq 2\delta$ and $j \geq 3$, then

$$\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j} > 0.$$

Proof of Claim 6.1.1. We show, equivalently, that

$$\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} > \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j}.$$

We notice that all terms of the sum on the left of the inequality are non-negative, as binomial coefficients are non-negative by definition, and that there are at least two terms of this sum as $j \geq 3$. This means that

$$\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} \geq \binom{\delta}{1} \binom{n-2\delta-c}{j-1} = \delta \binom{n-2\delta-c}{j-1}.$$

That is, that our sum is at least as large as its first term, which we know is nonzero by the assumption that δ is a positive integer. Thus, it suffices to show that

$$\delta \binom{n-2\delta-c}{j-1} > \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j}.$$

Dividing both sides by δ , and expanding these binomial coefficients with the factorial formula for binomial coefficients, we see that this is equivalent to showing that

$$\frac{(n-2\delta-c)!}{(j-1)!(n-2\delta-c-j+1)!} > \frac{(n-2\delta-c)!}{(n-2\delta-c)j!(n-2\delta-c-j)!},$$

or simply that

$$(n-2\delta-c)j!(n-2\delta-c-j)! > (j-1)!(n-2\delta-c-j+1)!.$$

Dividing both sides by $(j-1)!(n-2\delta-c-j)!$, we see that this is equivalent to the statement, $(n-2\delta-c)j > (n-2\delta-c-j+1)$. Since $j \geq 3$, to show this, it suffices to show that $(n-2\delta-c) > (n-2\delta-c-2)$, which is surely the case. \square

It follows immediately from this claim that

$$\sum_{j=3}^{t-1} \binom{c}{t-j} \left[\sum_{l=1}^{j-1} \binom{\delta}{l} \binom{n-2\delta-c}{j-l} - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{j} \right] > 0,$$

and thus it only remains to show that

$$-\binom{c}{t-1}\delta + \binom{c}{t-2} \left[\delta(n-2\delta-c) - \frac{\delta}{n-2\delta-c} \binom{n-2\delta-c}{2} \right] \geq 0.$$

Factoring out a δ and noticing that, by using the factorial formula for binomial coefficients and an argument similar to that used in the proof of Proposition 6.1.2,

$$\binom{c}{t-2} = \frac{c!}{(t-2)!(c-t+2)!} = \left(\frac{t-2}{c-t+2} \right) \frac{c!}{(t-1)!(c-t+1)!} = \frac{t-2}{c-t+2} \binom{c}{t-1},$$

we see that this is equivalent to showing that

$$\delta \binom{c}{t-1} \left[\frac{t-1}{c-t+2} \binom{n-2\delta-c+1}{2} - 1 \right] \geq 0.$$

If $c \leq t-1$, this surely holds as the binomial coefficient vanishes, and so we may assume that $c \geq t-1$, and show that

$$\frac{t-1}{c-t+2} \binom{n-2\delta-c+1}{2} - 1 \geq 0,$$

or equivalently, that

$$\frac{(t-1)(n-2\delta-c+1)}{2(c-t+2)} \geq 1, \quad \text{or} \quad (t-1)(n-2\delta-c+1) \geq 2(c-t+2).$$

Since, by construction, $2c \leq n-2\delta$, $n-2\delta-c+1 \geq c+1$. Also, since $t \geq 3$, $c-t+2 \leq c-1$ and $t-1 \geq 2$. This, to show this inequality, it suffices to show that $2(c+1) \geq 2(c-1)$. Since c is non-negative, this is surely the case. \square

Remark 6.1.2. We explained in Chapter 4, right before stating our reformulated conjecture, that we required that $t \geq 3$ because we were able to find counterexamples (in fact, infinitely many, of arbitrarily large n for $\delta = 2$, infinitely many of which are bipartite) for $t = 2$. As we will explain in the Chapter 6.2, this is the case for every δ . This proof actually reflects this point in a very clear way. If we allowed $t = 2$, then in equation (5),

$$i_2(K_{\delta,n-\delta}) - i_2(G) \geq c \left[(n-\delta-c) - \delta - \frac{n-\delta-c}{n-2\delta-c} (n-2\delta-c) \right] = -c\delta,$$

would hold, which makes sense, for any graph with less edges than $K_{\delta,n-\delta}$ should have more independent sets of size two (as discussed in, for example, Proposition 3.1.1). In particular, for any $c > 0$, if the $(n-\delta-c)$ -size partite set of G (which is smaller than the largest partite set size of $K_{\delta,n-\delta}$) has all its vertices minimum degree, then

G will have less edges, and thus, more independent sets. Specifically,

$$\begin{aligned} i_2(K_{\delta, n-\delta}) - i_2(G) &= \left[\binom{n}{2} - e(K_{\delta, n-\delta}) \right] - \left[\binom{n}{2} - e(G) \right] \\ &= e(G) - e(K_{\delta, n-\delta}) \\ &= (n - \delta - c)\delta - (n - \delta)\delta = -c\delta, \end{aligned}$$

as we expect from the proof. Thus, our sharpness is clearly depicted in the calculations.

6.2 On the sharpness of our main result

Our main result states that for any bipartite graph G of minimum degree at least δ , provided that $n \geq 2\delta$, $i_t(G) \leq i_t(K_{\delta, n-\delta})$ for any $t \neq 2$. This is a pretty strange condition, and it is one that we have briefly discussed thus far. In Chapter 5.3, we explained that among all graphs of minimum degree at least two, for any $n \geq 4$, C_n has less edges than $K_{\delta, n-\delta}$ and thus more independent sets of size two. Further, C_n is bipartite for any even n , as can be seen by labeling the vertices $1, 2, \dots, n$ such that consecutive integers are adjacent, and noticing that the integers of the same parity form partite sets. So, we have provided infinitely many counterexamples at $t = 2$ for $\delta = 2$. In the previous section, we even showed where part of our proof would not hold if t wasn't assumed to be at least three (keeping in mind that all graphs have the same number of independent sets of size one and two).

While these things do show sharpness of our theorem, it turns out that the theorem is much sharper than even these small results imply. For any n and any δ such that $n > 2\delta$, we will show that we can construct a bipartite graph on n vertices of minimum degree at least δ that has more independent sets of size two than $K_{\delta, n-\delta}$. Further, when $n = 2\delta$, we will show that $K_{\delta, n-\delta}$ has more independent sets of size two in Chapter 6.4, thus showing that there is no hope in sharpening any part of our result.

Proposition 6.2.1. *For any n and δ satisfying $n > 2\delta$, there is a bipartite graph G such that $i_2(G) > i_2(K_{\delta, n-\delta})$.*

Proof. This proof is an algorithmic one, meaning that we explain how to create the desired graph. It will involve modular arithmetic, floors and ceilings.

We briefly explain the idea of the proof here before beginning it. To create the desired bipartite graph, we place $\lceil \frac{n}{2} \rceil$ vertices into one partite set and the remaining $\lfloor \frac{n}{2} \rfloor$ into the other. We then make exactly δ edges adjacent to each vertex in the larger partite set, and spread them out as much as possible in the other partite set, leaving those vertices with minimum degree at least δ as well. This will create a bipartite graph with exactly $\lceil \frac{n}{2} \rceil \cdot \delta$ edges, which we will show is less than the number of edges in $K_{\delta, n-\delta}$ when $n > 2\delta$, which ensures that $K_{\delta, n-\delta}$ has less independent sets of size 2 by our earlier observation that for any graph G , $i_2(G) = \binom{n}{2} - e(G)$ (see Proposition 3.1.1).

Given n and δ satisfying the given requirements, we begin construction of our desired bipartite graph by $\lceil \frac{n}{2} \rceil$ vertices into one partite set, labeled $v_1, v_2, \dots, v_{\lceil \frac{n}{2} \rceil}$, and the remaining $\lfloor \frac{n}{2} \rfloor$ into the other, labeled $w_1, w_2, \dots, w_{\lfloor \frac{n}{2} \rfloor}$. For the remainder of this proof, when we write \bar{i} to represent the unique integer in $\mathbb{Z}_{\lfloor \frac{n}{2} \rfloor}$ which i is congruent to modulo $\lfloor \frac{n}{2} \rfloor$.

Make v_1 adjacent to every vertex of $\{w_i : i \equiv_{\lfloor \frac{n}{2} \rfloor} j \text{ for any } j \in \{1, \dots, \delta\}\}$, v_2 adjacent to every vertex of $\{w_i : i \equiv_{\lfloor \frac{n}{2} \rfloor} j \text{ for any } j \in \{\delta + 1, \dots, 2\delta + 1\}\}$, \dots , v_k adjacent

to every vertex of $\left\{w_i : i \equiv \lfloor \frac{n}{2} \rfloor j \text{ for any } j \in \{(k-1)\delta + (k-1), \dots, k\delta + (k-1)\}\right\}$. That is, we make v_1 adjacent to the first δ , v_2 adjacent to the next δ , etcetera, looping around back to w_1 when appropriate.

It is clear that this is a graph of minimum degree at least δ , and that, in fact, if n is not even then there are vertices of the $\lfloor \frac{n}{2} \rfloor$ -size partite set which have degree greater than δ . We can count the total number of edges in the graph by counting the number of edges adjacent to the $\lceil \frac{n}{2} \rceil$. By construction, there are δ edges adjacent to each v_i , and so the total number of edges in this graph is $\lceil \frac{n}{2} \rceil \delta$. The total number of edges in $K_{\delta, n-\delta}$ is $(n-\delta)\delta$. Thus, if n is even,

$$e(G) = \frac{n\delta}{2} \leq (n-\delta)\delta = K_{\delta, n-\delta},$$

as $n \geq 2\delta(G)$ by assumption, implying that $\frac{n}{2} \leq (n-\delta)$. We notice, as desired, that equality holds only in the case that $n = 2\delta$ which we handle later. If n is odd, then we do have the greater bound $e(G) = \frac{(n+1)\delta}{2}$, but we also have in this case that $n \geq 2\delta + 1$. Thus, $\frac{n}{2} \geq \delta + 1$, and so $n - \delta \geq \delta(G) + 1$, and so $e(G) = \frac{(n+1)\delta}{2} \leq K_{\delta, n-\delta}$, so this holds in this case as well, with a similar observation concerning equality. \square

6.3 Corollaries of our main result

The main corollary of our work is a bipartite version of Galvin's original Conjecture 5.2.1.

Corollary 6.3.1 (Alexander, Cutler and Mink). *Suppose n and δ are positive integers with $n \geq 2\delta$. If G is an n -vertex bipartite graph with minimum degree at least δ , then*

$$i(G) \leq i(K_{\delta, n-\delta}),$$

with equality if and only if $G = K_{\delta, n-\delta}$.

Before we prove this, however, we will prove a porism of our theorem that relates to the independent polynomial, which was the main focus of Chapter 4.

Porism 6.3.2. *If G is an n -vertex bipartite graph with minimum degree at least δ where $n \geq 2\delta$, then $P(G, x) \leq P(K_{\delta, n-\delta}, x)$ for all $x \geq 1$ with equality if and only if $G = K_{\delta, n-\delta}$.*

Proof. Our goal is to show that

$$P(K_{\delta, n-\delta}) - P(G, x) = \sum_{t=0}^{\infty} (i_t(K_{\delta, n-\delta}) - i_t(G))x^t \geq 0.$$

We look at the proof of our main result, Theorem 6.1.1, particularly at equation (5). Before inserting this rather large equation into the above, we simply things a bit by first recalling that $i_0(K_{\delta, n-\delta}) = 1 = i_0(G)$ and $i_1(G) = n = i_1(K_{\delta, n-\delta})$ (as proved in Proposition 3.1.1), and so

$$\sum_{t=0}^{\infty} (i_t(K_{\delta, n-\delta}) - i_t(G))x^t = \sum_{t=2}^{\infty} (i_t(K_{\delta, n-\delta}) - i_t(G))x^t.$$

We can also apply the computations (and explanation) of Remark 6.1.2 which immediately followed the proof and bounded our $t = 2$ term, to obtain the inequality

$$P(K_{\delta, n-\delta}) - P(G, x) \geq -c\delta x^2 + \sum_{t=3}^{\infty} \left[-\delta \binom{c}{t-1} + \delta \binom{c}{t-2} \frac{n-2\delta-c+1}{2} \right] x^t.$$

If we rewrite the term $-c\delta x^2$ in the more convenient form

$$-c\delta x^2 = \sum_{t=2}^2 -\delta \binom{c}{t-1} x^t,$$

and distribute the sum over the second term, we see that this lower bound is equivalent

to

$$\sum_{t=2}^2 -\delta \binom{c}{t-1} x^t + \sum_{t=3}^{\infty} -\delta \binom{c}{t-1} x^t + \sum_{t=3}^{\infty} \left[\delta \binom{c}{t-2} \frac{n-2\delta-c+1}{2} \right] x^t.$$

By combining the first two terms into one sum, this is equal to

$$\sum_{t=2}^{\infty} -\delta \binom{c}{t-1} x^t + \sum_{t=3}^{\infty} \delta \frac{n-2\delta-c+1}{2} \binom{c}{t-2} x^t.$$

We can re-index the sums to obtain the form

$$\sum_{t=2}^{\infty} -\delta \binom{c}{t-1} x^t + \sum_{t=2}^{\infty} \delta \frac{n-2\delta-c+1}{2} \binom{c}{t-1} x^{t+1}$$

for our lower bound. Factoring out the sum and the common $\delta \binom{c}{t-1}$ term, this gives us that

$$P(K_{\delta, n-\delta}) - P(G, x) \geq \sum_{t=2}^{\infty} \delta \binom{c}{t-1} \left(-1 + \frac{n-2\delta-c+1}{2} x \right) x^t.$$

If $c = 0$, then $\binom{c}{t-1} = 0$, so each term in the sum is zero. But, as discussed in the proof of Theorem 6.1.1 (to which this is a porism), if $c = 0$, then the graph we are looking at is $K_{\delta, n-\delta}$. If $c \geq 1$, then each coefficient is at least $\delta \binom{c}{t-1} (-1 + \frac{c+1}{2} x)$ as $c \leq \frac{n-2\delta}{2}$, which is clearly non-negative when $c, x \geq 1$. Thus, for any G in our class such that $G \neq K_{\delta, n-\delta}$, $P(K_{\delta, n-\delta}) - P(G, x) > 0$ when $x \geq 1$. \square

It is now easy to see why we delayed the proof Corollary 6.3.1. The proof follows always immediately from Porism 6.3.2.

Proof of Corollary 6.3.1. We recall from our discussion in Chapter 4, the section focused on the independence polynomial, that for any graph G , $P(G, 1) = i(G)$ (as discussed, specifically, in Remark 4.1.1). Thus, if we let $x = 1$ in the Porism 6.3.2, we immediately have that $i(G) \leq i(K_{\delta, n-\delta})$ provided that G satisfies the given restrictions. \square

We note that this corollary tells us, in particular, that even though $K_{\delta, n-\delta}$ does not always maximize $i_2(G)$ among n -vertex graphs of minimum degree at least δ , the maximization of this quantity for some G is not enough to cause a maximization of $i(G) = \sum_t i_t(G)$ as a whole. That is, even though $K_{\delta, n-\delta}$ may very well have less independent sets of size 2, it has the greatest total number of independent sets.

6.4 Other original results

In addition to the bipartite results presented so far, we offer two results about general (not necessarily bipartite) graphs that hold specifically for 2δ -vertex graphs. They are not particularly strong results on their own, but together with the other original results presented thus far, they do provide strong evidence for Conjecture 5.3.2. Besides for extending our results to non-bipartite graphs in this case, these results provide even stronger results in that our bounds hold here for $t = 2$ as well, the equality that previously failed for infinitely many values of n (see Remark 6.1.2).

Theorem 6.4.1. *If G is any 2δ -vertex graph with minimum degree at least δ , then*

$$i_t(G) \leq i_t(K_{\delta, n-\delta}) = i_t(K_{\delta, n-\delta}) = i_t(K_{\delta, \delta})$$

for all $t \geq 0$.

Proof. We show this by induction on t . We have that $i_1(G) = n = i_1(K_{\delta, \delta})$ by Proposition 3.1.1. Assume that $i_t(G) \leq i_t(K_{\delta, \delta})$. We show that $i_{t+1}(G) \leq i_{t+1}(K_{\delta, \delta})$. Let $\mathcal{J}_{t+1}(G) := \{(v, I_t) : v \notin N(I_t) \cup I_t, I_t \in \mathcal{I}_t(G)\}$. Informally, each $(v, I_t) \in \mathcal{J}_{t+1}(G)$ is a pair which contains an independent set of size t in G , together with a vertex that is not a vertex in I_t , and which is not adjacent to any vertex of I_t . We notice that, in this case $I_t \cup \{v\}$ must form an independent set, and specifically, $I_t \cup \{v\} \in \mathcal{I}_{t+1}(G)$. We notice that each $I_{t+1} \in \mathcal{I}_{t+1}$ will appear in exactly $t+1$ pairs in $\mathcal{J}_{t+1}(G)$. This is because, for any vertex $v \in I_{t+1}$, $I_{t+1} - \{v\}$ is an independent set which v is not contained in, and which v is not adjacent to any vertex of. This means that

$$|\mathcal{J}_{t+1}(G)| = (t+1) |\mathcal{I}_t(G)| = (t+1) \cdot i_t(G). \quad (6.6)$$

Further, we notice that for every $I_t \in \mathcal{I}_t(G)$, $N(I_t) \geq \delta$ since G has minimum degree at least δ by assumption, and also that $N(I_t) \cap I_t = \emptyset$ by definition. Thus, there can be at most $n - |N(I_t) \cup I_t| \leq n - \delta - t$ pairs of the form $(I_t, v) \in \mathcal{J}_{t+1}(G)$ for any given $I_t \in \mathcal{I}_t(G)$. It follows that

$$|\mathcal{J}_{t+1}(G)| \leq |\mathcal{I}_t(G)| (n - \delta - t) = (n - \delta - t) i_t(G).$$

Putting this together with equation (6), we have that

$$i_{t+1}(G) = \frac{1}{t+1} |\mathcal{J}_{t+1}(G)| \leq \frac{(n - \delta - t)}{t+1} i_t(G).$$

We note that for any $I \in \mathcal{I}_t(K_{\delta, \delta})$, $N(I_t)$ is exactly the partite set of $K_{\delta, \delta}$ which it is not contained in, and so $K_{\delta, \delta}$ is a graph which obtains equality in the above equation. Thus,

$$i_{t+1}(G) \leq \frac{n - \delta - t}{t+1} i_t(G) \leq \frac{n - \delta - t}{t+1} i_t(K_{\delta, \delta}) = i_{t+1}(K_{\delta, \delta}),$$

where the second inequality is by our induction assumption. □

As this theorem states that for any graph G satisfying our requirements, $i_t(G) \leq i_t(K_{\delta, n-\delta}) = i_t(K_{\delta, \delta})$ for every t we consider, and $i(G) = \sum_t i_t(G)$, it is immediate that $i(G) \leq i(K_{\delta, \delta})$ holds as well. Thus, we have the following.

Corollary 6.4.2. *If G is any 2δ -vertex graph with minimum degree at least δ , then*

$$i(G) \leq i(K_{\delta, \delta}).$$

Also, as the coefficients of the independence polynomial are exactly the terms bounded in our theorem, we also have this next corollary immediately.

Corollary 6.4.3. *If G is any 2δ -vertex graph with minimum degree at least δ , then*

$$P(G, x) \leq P(K_{\delta, \delta}, x)$$

for any $x \geq 0$.

Chapter 7

Implications of our original results

We have stated all original results of this work in the past few sections, their motivation, and we have provided proofs of them. We now discuss some implications of these results other than those which directly motivated them, and we explain how these results can be used as building blocks for future work.

7.1 Relationships to work of the past

We now show, as discussed previously, that Prodinger and Tichy's original 1982 upper bound result on $i(G)$, that the star maximizes the total number of independent sets among trees, and thus all connected graphs, can be viewed as a corollary of Theorem 6.1.1, and a very well known result. This result is one of the first major results of graph theory, and it was stated and proven by one of the pioneers of the subject, Dénes Kőnig in 1931, [18]. It is the oldest theorem that we are presenting in this paper.

Theorem 7.1.1 (Kőnig 1931). *A graph is bipartite if and only if it contains no odd cycles.*

As mentioned while we were studying unicyclic graphs in Chapter 2.1, there is some strong connection between the lengths of cycles in graphs and the structure of their independent sets. This theorem tells us that the vertex set of a graph can be partitioned into two disjoint independent sets if and only if there are no cycles of odd length in the graph. One implication of the theorem, that if a graph is bipartite then it contains no odd cycles, comes to one with little thought. Because no two vertices in the same partite set can be adjacent, any cycle must start and end in the same partite set of a bipartite graph, and must always move to the other partite set as we alternate vertices, thus transversing an even number of edges. The converse of this statement, however, is surprising at first glance. For such a powerful theorem, the proof is actually quite elementary, but does require a few preliminary results.

Because the proof is not particularly relevant to what we are studying, we will not present it here. However, we will restate, and show how easy it is to prove Prodinger and Tichy's result (which took a somewhat robust induction argument before) using these two theorems.

Theorem (Prodinger and Tichy 1982). For any tree T on n vertices,

$$i(t) \leq i(K_{1,n-1}),$$

thus implying that $K_{1,n-1}$ has the most number of independent sets among all bipartite graphs.

Proof. Consider any tree T . As a tree contains no odd cycles, by König's Theorem, T is bipartite. Also, as trees are connected by definition, $\delta(T) \geq 1$, and so, by the corollary of our main result (Theorem 6.3.1), $i_t(G) \leq i_t(K_{1,n-1})$. \square

In the same way that Cutler and Radcliffe's result about fixed number of edges (Theorem 3.3.2) can be viewed as a generalization of many other results that we have discussed here, so can Theorem 6.1.1 for obvious reasons (such as the one just presented). In light of König's theorem, for any graph G which does not contain odd cycles, $i(G)$, $i_t(G)$, and $P(G, x)$ can all be bounded by simply looking at the graphs minimum degree. Also, any graph from which we can obtain a bipartite graph by deleting edges can be bounded similarly.

7.2 A justified conjecture

Our stronger conjecture, which is a reformulation of the conjecture of Galvin discussed in Chapter 5.3, is still unproven. We end this work with a restatement of this conjecture, for which we credit Galvin. Galvin's original 2011 conjecture, Conjecture 5.2.1, which he proved asymptotically also remains unproven. Most likely, as was the case with the bipartite version, a proof of the stronger conjecture would leave Galvin's a corollary, but this remains to be seen. Every result mentioned in sections 6.1 to 6.4 is another piece of evidence that these conjectures are true; and the more work that we do on independent sets, the more we do believe that they are. Though the proof currently eludes us, we do believe that we will one day see it in print, and know for absolute certain that it is true, whether it be an original proof of ours, one that uses ours as a stepping stone the way Zhao used Kahn's for regular graphs (as discussed in Chapter 5.1), or one that is completely different.

Conjecture (Galvin). Let n , δ , and t be positive integers with $n \geq 2\delta$ and $t \geq 3$. If G is any graph on n vertices such that $\delta(G) \geq \delta$, then

$$i_t(G) \leq i_t(K_{\delta, n-\delta}),$$

with equality if and only if $G = K_{\delta, n-\delta}$.

Bibliography

- [1] B. Bollobás, *Modern Graph Theory*, ser. Graduate Texts in Advanced Mathematics. Springer Publishing Company, 1998.
- [2] H. Prodinger and R. Tichy, "Fibonacci numbers of graphs," *The Fibonacci Quarterly*, vol. 20, pp. 16–21, 1982.
- [3] E. Merrifield and H. Simmons, "Topological methods in chemistry," *Discrete Mathematics*, 1989.
- [4] A. Pedersen and P. Vestergaard, "Bounds on the number of vertex independent sets in a graph," *Taiwanese Journal of Mathematics*, vol. 10, pp. 1575–1587, 2006.
- [5] T. Tao and V. Vu, *Additive Combinatorics*, ser. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2006.
- [6] J. Cutler and A. Radcliffe, "Extremal graphs for homomorphisms," *Journal of Graph Theory*, vol. 67, p. 261284, 2011.
- [7] H. S. Wilf, *Generatingfunctionology*, ser. Second Edition. California: Academic Press, 1993.
- [8] I. Gutman and F. Harary, "Generalizations of the matching polynomial," *Utilitas Mathematica*, vol. 24, pp. 97–106, 1983.
- [9] J. L. Arocha, "Propiedades del polinomio independiente de un grafo," *Revista Ciencias Matematicas*, vol. 5, pp. 103–110, 1984.
- [10] P. Matchett, "Operations on well-covered graphs and the roller-coaster conjecture," *Electronic Journal of Combinatorics*, vol. 11, p. 45, 2004.
- [11] Y. Alvai, P. J. Malde, A. J. Schwenk, and P. Erdős, "The vertex independence sequence of a graph is not constrained," *Congressus Numerantium*, vol. 58, pp. 15–23, 1987.
- [12] E. A. Bender and E. R. Canfield, "Log-concavity and related properties of the cycle index polynomials," *Journal of Combinatorial Theory*, vol. A74, pp. 57–70, 1996.

- [13] V. Levit and E. Mandrescu, "On well-covered trees with unimodal independence polynomials," *Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing*, vol. 159, p. 193202, 2002.
- [14] J. Kahn, "An entropy approach to the hard-core model on bipartite graphs," *Combin. Probab. Comput.*, vol. 10, no. 3, pp. 219–237, 2001.
- [15] Y. Zhao, "The number of independent sets in a regular graph," *Combin. Probab. Comput.*, vol. 19, no. 2, pp. 315–320, 2010. [Online]. Available: <http://dx.doi.org/10.1017/S0963548309990538>
- [16] D. Galvin, "Two problems on independent sets in graphs," *Discrete Math.*, vol. 311, no. 20, pp. 2105–2112, 2011.
- [17] H. Hua, "A sharp upper bound for the number of stable sets in graphs with given number of cut edges," *Applied Mathematics Letters*, vol. 22, pp. 1380–1385, 2009.
- [18] D. Kőnig, "Gráfok és mátrixok," *Matematikai s Fizikai*, vol. 38, pp. 116–119, 1931.
- [19] H. B. Enderton, *Elements of Set Theory*. United States of America: Academic Press, 1977.
- [20] D. Smith, M. Eggen, and R. St. Andre, *A Transition to Advanced Mathematics*. Boston: Cengage Learning, 2011.
- [21] R. P. Stanley, *Enumerative Combinatorics: Volume 1*, ser. Cambridge University Press. Cambridge: Cambridge Studies in Advanced Mathematics, 1999.
- [22] T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of the Combinatorial Proof*. United States of America: The Mathematical Association of America, 2003.

Appendix A

Basic set theory

Throughout these appendices, we will introduce many mathematical concepts, most of which are common throughout mathematics, not just in graph theory. We will, however, be looking at almost all of these concepts exclusively through graph theory lenses. That is, we will explain only how to use them in graph theory, for this is the only way they will be used in this book.

A.1 Definition of a set and explanation of the definition of a graph

A set is a collection of things. The things in a set are called its *elements*. We denote a set with curly braces that surround the elements of that set. If we want to write a set with many elements in it, but don't want to write them all out, we often use three dots to mean, informally, "put in the obvious elements we are not mentioning." For example, if we want to say that A is the set of all integers from 0 to 10, we may write,

$$A = \{0, 1, 2, \dots, 10\}.$$

In fact, if we wanted to be even more technical, we would write $A := \{0, 1, 2, \dots, 10\}$, because in mathematics we use the $:=$ symbol to mean, "is defined to be." In this case, we are giving A a definition; we were defining it to be the integers from 1 to 10. This is notation that we will commonly use.

If we have a set A and an object a , we use the notation $a \in A$ to mean that a is an element of A , and use the notation $a \notin A$ to mean that a is not an element of A . With A defined as above, we see that $0 \in A$, $1 \in A$, $2 \in A$, etc., but that, for example, $100 \notin A$. We use \emptyset , the so called *empty set*, to denote the set with no elements in it. That is, $\emptyset := \{ \}$, an empty container. We can say, right from this definition, that for any possible a , $a \notin \emptyset$. If we want to make it clear that some set is not the empty set, that is, that it does have things in it, we refer to that set as being

nonempty. The set A for example, as defined above, is a nonempty set.

We say that a set B is a *subset* of A if every element in B is also in A . We use the notation $B \subseteq A$ to denote this. Putting all this notation together, we say that $B \subseteq A$ if for every $b \in B$, we have that $b \in A$. If we take A to be $\{0, 1, 2, \dots, 10\}$ as we did before, and B to be the set of all odd integers from 1 to 10, that is, $B := \{1, 3, 5, 7, 9\}$, then $B \subseteq A$. However, if we define $B := \{5, 10, 15, 20\}$, B is not a subset of A .

Remark A.1.1. We do not allow sets to contain duplicated elements. This means that the set $\{1, 1, 2\}$ is the same as the set $\{1, 2\}$, is the same as the set $\{1, 1, 1, 1, 1, 2\}$. This will prove to be an important point when we prove Proposition B.3.1 in the next Appendix.

In Chapter 1.1, we define a graph G as a mathematical structure consisting of two sets. One is a nonempty set of elements called vertices, which we denote $V(G)$, and the other is a set we denote $E(G)$, of 2-element subsets of the vertices that we call edges. By 2-element subsets, we intuitively mean subsets that have two elements. This means that if $V(G)$ is a set with n vertices in it, say $V(G) = \{v_1, v_2, \dots, v_n\}$, then $E(G)$ can have elements like $\{v_1, v_3\}$, or $\{v_5, v_6\}$, or any $\{v_i, v_j\}$ provided that i and j are some distinct integers between 1 and n , because these are 2-element subsets of $V(G)$.

For those without a mathematical background, this can seem like a strange definition. Visual aids, however, can make things much clearer.

A.2 How to draw a graph

We draw a visual representation of a graph as follows: Given a graph G , for any vertex in $V(G)$, we draw a small circle and label it with whatever we have called the vertex. If, for two vertices, say u and v , $\{u, v\} \in E(G)$, draw a line between u and v . We provide a specific example.

Example A.2.1. If we want to draw a graph G that has $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$, we would start by drawing four small circles, and labeling one of them v_1 , another v_2 , etc. for all four of them. We would then draw a line from circle v_1 to circle v_2 , a line from circle v_2 to circle v_3 , and a line from circle v_3 to circle v_4 . There are many ways that this picture can look. When we draw the vertices, we could have drawn them in a straight line, or scattered in no particular shape. We also could have labeled the vertices by writing their labels inside the circles, we could have filled in the circles and written labels on the outside, etc. It really doesn't matter, as long as we capture all of the information. One way that this picture can look is as in Figure A.1. This type of graph (that forms a line), is actually a special kind of graph we call a path. Paths are introduced in detail in Chapter 1.1. As can be seen in that section, we denote a path on n vertices by P_n . Since this is a path on 4 vertices, we would call this picture a P_4 .



Figure A.1: The path P_4

If $\{u, v\} \in E(G)$ for some G , then we say that $\{u, v\}$ is an edge between u and v , or simply that u and v are adjacent, for reasons that can be deduced from the picture. Usually, for convenience, we just write uv to represent $\{u, v\} \in E(G)$. That is, we drop the curly braces when there is no risk of confusion.

A.3 Related notation and basic set operations

When defining a set previously, we represented a set A of all integers from 0 to 10 by $A = \{0, 1, 2, \dots, 10\}$. While this is perfectly valid, this notation is not always practical. If, say, we had a graph G on 1,000 vertices, and we wanted to talk about all subsets of size 100 (that is, sets contained in $V(G)$ which contain 100 elements), we would not want to list out all these thousands of elements. Instead, we use what is called a *conditional definition of a set*. In set theory, $:$, is a symbol used to mean “such that.” Another common symbol is \mathbb{N} , which means the set of all (infinitely many) integers greater than or equal to zero, $\{0, 1, 2, \dots\}$. Using this, the less than or equal to symbol \leq , and the $:$, we can write $A := \{0, 1, 2, \dots, 10\}$ as $A := \{a \in \mathbb{N} : a \leq 10\}$. It is the set of all elements a in \mathbb{N} such that a is at most ten. That is exactly the set $\{0, 1, 2, \dots, 10\}$. That is, $\{a \in \mathbb{N} : a \leq 10\}$ and $\{0, 1, 2, \dots, 10\}$ are two ways of writing the same thing. In Chapter 1.1, we define the neighborhood of a vertex as the set of all vertices to which it is adjacent, using this type of notation. We say that given a vertex v , the neighborhood of v is

$$\{u \in V(G) : \{u, v\} \in E(G)\}.$$

We see with our newfound notation that this exactly means, the set of all vertices $u \in V(G)$ such that uv is an edge. That is, the set of all vertices of $V(G)$ that are adjacent to v .

We often want to talk about the number of elements in a set A . For this, we use the notation $|A|$, and refer to this as the *cardinality* of A , or simply the *size* of A . So, if $A := \{0, 1, 2, \dots, 10\}$, then $|A| = 11$. As explained in Chapter 1.1, the number of vertices in a graph G , or equivalently, the size of $V(G)$, is denoted $n(G)$. Using this notation, $n(G) := |V(G)|$. The number of edges in a graph G is denoted $e(G) := |E(G)|$. In the graph G defined in Example A.2.1, we see that $n(G) = 4$ and $e(G) = 3$.

Most of modern graph theory is centered around the comparison of different graphs, and almost always, we only compare graphs that have the same number of vertices. This is because we are, most often, trying to answer questions that are concerned with the most efficient ways to connect some given objects. This very

vague statement implies that we are usually given a certain number of vertices, say n many, and we are asked how to place the edges so that the graph has the most number of something, or the least number of something else. The two most extreme types of graphs we can make, are graphs that have all possible edges, and graphs that have no edges. As explained in Chapter 1.1, we call a graph complete if it has all possible edges, and empty if it has none. We denote the complete graph on n vertices K_n , and the empty graph on n vertices E_n . On four vertices, for example, K_4 and E_4 look like the graphs in Figure A.2.

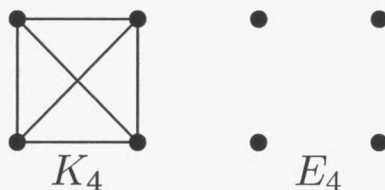


Figure A.2: The complete graph K_4 and the empty graph E_4

In Chapter 1.1, we also define the degree of a vertex as the number of things in its neighborhood. That is, given a vertex v , we define the degree of $v \in V(G)$, which we denote $d_G(v)$, as $d_G(v) := |N_G(v)|$. In the graph G of Example A.2.1,

$$d_G(v_1) = 1, \quad d_G(v_2) = 2, \quad d_G(v_3) = 2 \quad \text{and} \quad d_G(v_4) = 1.$$

When we place the words *max* or *min* in front of a set, we are saying: take the biggest or smallest thing in that set, respectively. Below the words max or min, when we are using them this way, we specify which elements of which set we are looking at. This type of indexing is discussed in more detail in the next section of this appendix. For $A = \{1, 2, \dots, 10\}$, for example,

$$\max_{a \in A} \{2 \cdot a\} = 2 \cdot 10 = 20 \quad \text{and} \quad \min_{a \in A} \{2 \cdot a\} = 2 \cdot 1 = 2,$$

because $\max_{a \in A} \{2 \cdot a\}$ states that we should take the biggest thing $2 \cdot a$ can be when a is coming from A (denoted $a \in A$). The $\min_{a \in A}$ expression looks for the smallest.

In Chapter 1.1 we explain that $\max_{v \in V(G)} \{d_G(v)\}$ is referred to as the maximum degree of a graph G , because this is exactly how to define the largest degree of any vertex in that graph. In the graph G of example A.2.1, we see that the maximum degree is 2 (and similarly that the smallest degree, referred to as the minimum degree, is 1).

Sometimes we want to use certain operations on sets. Two operations we perform in this book are the *union* operation and the *intersection* operation. The union of two sets A and B , denoted $A \cup B$, is the set of all elements contained in A or B . The intersection of A and B , denoted, $A \cap B$ is the set of all elements that are in both sets. For example, if $A := \{1, 2, 3, 4\}$ and $B := \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$ and $A \cap B = \{3, 4\}$.

More often than not in mathematics, when we want to take the union of sets, it's of much more than two sets. For this, we index the set the set in a special way described in the next section, "Indexing."

A.4 Indexing

Sets have no order to them. That is, if we write the set $\{1, 2, 3\}$, that's the same as writing the set $\{2, 3, 1\}$, or the set $\{3, 1, 2\}$. There is no way of saying one element is first in the set, or second; there is no inherent listing. If we want to have a set of things that are given in some particular order, we write them as, what is called, a *sequence*.

We denote the sequence a_1, a_2, \dots , which is usually infinite and indexed by \mathbb{N} (meaning that all elements are labeled with subscripts that are 0 or a positive integer), and which we can think of as a list ordered by its indices, by $(a_t)_{t \in \mathbb{N}}$. For example, say we want to express the list $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, say infinitely long. We write this as

$$(a_t)_{t \in \mathbb{N}} \quad \text{where} \quad a_t := \frac{1}{t+1} \quad \text{for every} \quad t \in \mathbb{N}.$$

In this case, $a_0 = \frac{1}{1+0} = 1$, $a_{100} = \frac{1}{100}$, etc. The $t \in \mathbb{N}$ tells us that our elements are indexed, or labeled, by elements of \mathbb{N} , and our definition tells us what those infinitely many elements are. Though it is most common, sequences need not be infinite. For example,

$$(a_t)_{t \in \{1, 2, 3\}} \quad \text{where} \quad a_t := \frac{1}{t+1} \quad \text{for every} \quad t \in \mathbb{N}.$$

is simply the 3 element list $a_1 = \frac{1}{2}, a_2 = \frac{1}{3}, a_3 = \frac{1}{4}$. The discussions of the following two mathematical structures will aid in the understanding of this seemingly strange indexing.

A piece of notation that is seen quickly in Chapter 1.1, and which is incredibly common throughout this whole work (and all of mathematics for that matter), is the symbol Σ . This is a symbol that simply means add a bunch of things up. *Which* things we are adding up is determined by the things written below (and sometimes above) the Σ , with an indexing style similar to that of a sequence. For example, if we still consider set A with the definition $\{1, 2, \dots, 10\}$, then, $\sum_{a \in A} 2 \cdot a$ tells us to add up $2 \cdot a$ for every element a in A . That is,

$$\sum_{a \in A} 2 \cdot a = 2 \cdot 1 + 2 \cdot 2 + \dots + 2 \cdot 9 + 2 \cdot 10 = 2 + 4 + 6 + 8 + \dots + 18 + 20.$$

In Chapter 1.1, the first theorem that we consider involves the sum

$$\sum_{v \in V(G)} d_G(v).$$

Thus sum tells us to add up the degrees of every different vertex in $V(G)$ for some given graph G . In the graph G defined in Example A.2.1,

$$\sum_{v \in V(G)} d_G(v) = d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4) = 6.$$

We can use similar indexing for unions. Say we have arbitrarily many sets, call them A_1, A_2, \dots , etc., maybe infinitely many, and we want to take the union of all of them. We write this as $\cup_{i \in \mathbb{N}} A_i$. It means union the A_i 's where $i = 1, i = 2$, etc. The first place in the main body of this work that we use this symbol is in Chapter 1.2, when we mention something called independent sets. Before going into this example, it is worth talking about independent sets in some detail. As one can probably guess from the title of this book, independent sets will come up very often. An independent set is a set of vertices, no two of which are adjacent. That is, given a graph G , an independent set is a set $I \subseteq V(G)$ such that, if $u, v \in I$, then $uv \notin E(G)$. For example, if we let G be the graph in Figure 1.5 at the beginning of Section 1.2, then we see that the red vertices form an independent set of size three.

For any graph G , we let $\mathcal{I}_t(G)$ denote the set of all independent sets of size t in G for any $t \in \mathbb{N}$. So, if G is the graph defined by the above figure, we would say that the red vertices form a set which is an element of $\mathcal{I}_3(G)$. We now tie this back to the idea of unions. If we want to consider all independent sets in a graph, of any size, then we want to take the union of the sets of different sizes. That is, we want to have all the independent sets from all the different containers which were divided by size. We want all the independent sets of size one, of size two, etc. This is exactly $\cup_{t \in \mathbb{N}} \mathcal{I}_t(G)$.

We will finish this section by defining two final set theory terms, disjoint and partition. We say that two sets A and B are *disjoint* if they have no elements in common. For example, the sets $A = \{1, 2, \dots, 10\}$ and $B = \{100, 101, 102\}$ are disjoint, while the sets $C = \{1, 2, 3, 4\}$ and $D = \{3, 4, 5, 6\}$ are not. A partition of A , informally, is a set of subsets of A which have nothing in common, but which make up all of A . Formally, using the notation we've developed, a partition of A is a group of subsets of A , say A_1, A_2, \dots, A_k for some $k \in \mathbb{N}$, which are disjoint, such that $\cup_i A_i = A$ (that is, such that their union is the whole set A). In this case, we say that A is *partitioned* into the sets A_1, A_2, \dots, A_k . If $A = \{1, 2, \dots, 10\}$, then one example of partition of A is that made up of the sets $A_1 = \{0\}$, $A_2 = \{1, 2, 3, 4\}$, $A_3 = \{5, 6, 7\}$, and $A_4 = \{8, 9, 10\}$. This is because these sets have nothing in common (that is, they are disjoint), and their union, $\cup_i A_i$ is exactly A . A partitioning is simply a dividing up of some set into pieces.

In Chapter 1.1, we define what it means for a graph to be bipartite. We say that a graph G is *bipartite* if $V(G)$ can be partitioned into 2 sets, say A and B , such that no two vertices of A are adjacent, and no two vertices of B are adjacent. Using the terminology we've developed, we can equivalently say that a graph G is bipartite if $V(G)$ can be partitioned into two independent sets. For example, Figure A.3 depicts

a bipartite graph on six vertices. The vertices are partitioned into those lined up on

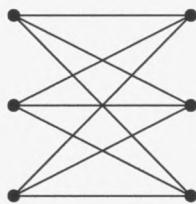


Figure A.3: A bipartite graph on six vertices

the far right of the figure, and those lined up on the far left. These sets are clearly independent.

We now have enough set theory notation to build off of for our purposes, but have only begun to skim the surface of set theory. For a more comprehensive introduction, see [19]. The next appendix deals with how to actually understand and prove mathematical statements.

Appendix B

Mathematical proofs

B.1 Introduction to mathematical proof

One of the most beautiful things about mathematics is that once a statement is proven, it can never be disproven. Once we know that something is true, it can never be argued that is false without breaking down and redefining the entire structure of the subject. The types of statements that mathematicians prove always boil down to some if-then form, or some compounding of if-then statements, even if they are far too complicated to seem like it. By this, we mean that all statements that mathematicians prove can be thought of as one or more statements of the form, “if P then Q ,” where P is some mathematical property, or set of mathematical properties, and Q is some other mathematical property. For example, the statement, “if n is an integer greater than ten, then n is an integer” (this statement, of course, being one that is true).

When talking about mathematical statements, one word that is very common is *converse*. The converse of a statement is its reverse. The converse of the statement, “if n is a an integer greater than ten, then n is an integer,” is the statement, “if n is an integer, then n is an integer greater than ten.” Even though the original statement was true, this converse of the statement is false, as n can be, say, 5. Mathematicians refer to a statement which is true, and which also has a true converse, as an *if and only if statement*, for obvious reasons, and they always write it with these words. An example of such a statement is, a number is a multiple of ten if and only if it ends in 0.

The most common names for mathematical statements that have been proven are either *theorem*, *proposition*, or *corollary*. We will not explain the difference between a theorem and proposition here, we just note that they are very similar. A corollary is something that follows almost immediately from some given theorem. A similar term to corollary is *porism*, which is a result that follows almost immediately from the proof of another (rather than the result itself). It is not recommended that the reader get caught up in the differences between these terms at this point, just know

that they refer to mathematical statements that have been proven. A *conjecture* is a mathematical statement that a mathematician believes is true, but which has not yet been proven.

There are many techniques for proving statements, essentially three of which we will use in this work, allowing for some variation, sometimes in combination with each other. The first is the most direct type of proof we consider.

B.2 Combinatorial Proofs

The first type of proof we look at is a proof obtained by direct counting. This type of proof is called a *combinatorial proof*, and it is a basis for the field of combinatorics, a field of mathematics that graph theory can be viewed as a subfield of. The next and final section of the appendix, C, is dedicated to this. However, we introduce the idea here by giving an example of a combinatorial proof, one that shows that two things are equal because they count the same thing. The statement that we prove is Theorem 1.1.1, which can be found in Chapter 1.1, and which we restate here.

Theorem (The First Theorem of Graph Theory). For any graph G ,

$$\sum_{v \in V(G)} d_G(v) = 2e(G).$$

In Chapter 1.1, we give the following proof of the theorem.

Proof. Our goal is to show that when you add up the degrees of every vertex in the graph, this counts the number of edges twice. This is the case, for when we are summing the degrees of the vertices of G , we count each edge twice, one for each vertex adjacent to it. \square

This can be incredibly unconvincing to someone unfamiliar with combinatorial proofs. To prove this theorem, what we are doing is showing that the quantity on the left side of the equation counts the same thing as the quantity on the right side. The quantity on the right side is clearly counting every edge of the graph twice, since $e(G)$ is the number of edges, and so $2e(G)$ is two times the number of edges. We thus need to show that the left also counts this. Specifically, the left is the sum of all degrees of the vertices in G . So, for each vertex, we are adding into the sum the number of edges adjacent to it. We need to show that this adding up of degrees is the same as counting each edge twice. The idea is that every edge is adjacent to exactly two vertices, so when we add up all these vertex by vertex, we add two times each edge.

Looking at the graph G given in example A.2.1 to aid in our understanding of

this proof, we see that when we compute

$$\sum_{v \in V(G)} d_G(v) = d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4) = 6,$$

we are essentially counting the edge v_1v_2 once when we add in $d_G(v_1)$ and once when we add in $d_G(v_2)$, and thus we count it exactly twice. Similarly for all other vertices and edges.

Remark B.2.1. We notice that at the end of every proof of this work, we place a little square. This is common practice throughout mathematics. It is a signal to the reader that we have shown what we set out to. That is, that the proof is complete.

B.3 Proof by Contradiction

The second type of proof we consider is a proof by contradiction. That is, we assume that what we want to prove is not true, and show this is nonsensical. For a silly example, let's say we want to prove that a cow is not a human. To do this proof by contradiction, we would assume a cow is a human. Then, since a cow is a human, it has two legs. However, a cow doesn't have two legs, it has four legs, a contradiction. So, if we assume a cow is a human we get a contradiction, thus, our assumption must have been faulty, and so we can deduce from this that a cow must not be a human. A nice example of a graph theoretic proof by contradiction is Theorem 1.1.2 in Chapter 1.1, but the previous parts of Chapter 1.1 will need to be read and understood before attempting this proof.

We look now at a very important proposition which we will prove by contradiction. It allows us to better capture the idea of a general graph. It tells us that no vertex can ever be adjacent to itself, and that there cannot be more than one edge between two vertices.

Proposition B.3.1. *For any graph G , and any vertices $u, v \in V(G)$, we have the following properties.*

1. $\{v, v\} \notin E(G)$.
2. $\{u, v\}$ cannot appear in $E(G)$ more than once.

Proof. We want to show that $\{v, v\} \notin E(G)$. Assume, to the contrary, that $\{v, v\} \in E(G)$. By remark A.1.1. $\{v, v\} = \{v\}$, and so it is a 1-element subset. This means that $E(G)$ contains a 1-element subset $V(G)$. However, by definition of a graph, $E(G)$ can only contain two element subsets, and so we have contradicted the definition of $E(G)$. It must be that our assumption is false, i.e., that $\{v, v\} \in E(G)$ cannot happen. We have shown that $\{v, v\} \notin E(G)$. This proves (1). To prove (2), assume to the contrary that $\{u, v\}$ appears in $E(G)$ more than once. Then, we have that

$E(G)$ is a set which repeats an element, but this directly contradicts Remark A.1.1. We have proven (2) by contradiction as well. \square

B.4 Proof by Induction

The final kind of proof we consider is the most complicated of the three, and it is called proof by mathematical induction. It's a little strange at first, but once you become comfortable with it, it feels very natural. Consider the following claim:

Claim B.4.1. If we add up the first n positive integers, then their sum is $\frac{n(n+1)}{2}$. Symbolically,

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

This claim asserts, for example, when $n = 3$, that $1 + 2 + 3 = \frac{3(3+1)}{2} = 6$. It is easy to see in this case that the claim holds. So, we have shown that the claim is true when $n = 3$. But how can we possibly prove that it's true for *all* n ? That's infinitely many numbers. Mathematicians use the method of mathematical induction for this. They first show that the statement is true for the smallest thing you want it to be, here $n = 1$, then they show that *if* it's true for all integers (within whatever range you are considering) *less* than or equal to some k , *then* it's true for k itself. So, in this case, we would show it's true for $n = 1$ first, and then prove the statement: "if it's true for $n \leq k$, it's true for $n = k + 1$."

Why is this a valid proof technique? Well, once we have proved it true for $n = 1$, and we've proved the statement: "if it's true for $n \leq k$, it's true for $n = k + 1$ ", then we look at $n = 2$ and reason as follows:

We know from the statement that if it's true for $n \leq 1$, that is, if it's true for $n = 1$, then it's true for $n = 2$. We proved it's true for $n = 1$, so we know that it's true for

$$n = 2$$

The logic continues upward. We want to know if it's true for $n = 3$. We directly shows that it was true for $n = 1$, we just argued that it was true for $n = 2$, and thus it is true for $n \leq 2$. We also proved the statement, "if it's true for $n \leq k$, it's true for $n = k + 1$," so, since it's true for $n \leq 2$, it's true for $n = 3$. It's a bit difficult to wrap your mind around at first, and it is oftentimes the first really point of struggle for students entering advanced mathematics. However, let's prove Claim B.4.1 by mathematical induction this way, then prove another statement, and this should help clear things up a bit.

Proof of Claim B.4.1. As stated, we need to first show it's true for the smallest n we want it to be true for, here, $n = 1$. As $1 = \frac{1(1+1)}{2} = 1$, the claim surely holds here. Now we need to prove that if the statement is true for some $n = k$, then the statement

is true for $n = k + 1$. So, assume that this statement is true for $n = k$, that is, that $1+2+3+\cdots+k = \frac{k(k+1)}{2}$. We need to show that with this assumption, we can prove that it's true for $n = k+1$, that is, that $1+2+3+\cdots+k+(k+1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+1)}{2}$. Well, we assumed that $1+2+3+\cdots+k = \frac{k(k+1)}{2}$, and so, using this:

$$1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1).$$

With some basic algebra we see that

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+1)}{2},$$

and so we have shown that $1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+1)}{2}$, as desired. The claim has been proven by mathematical induction. \square

For a more graph theoretic example, we prove a basic result about a type of graph called a path. A path on n vertices, denoted P_n is a graph that is discussed briefly in Chapter 1.1. It is of the form $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. It is essentially a "straight line" graph. The graph G defined in example A.2.1, as mentioned, is an example of P_4 . If we draw P_1 , we see that this must be a single vertex with no edges, if we draw P_2 we see that it is two vertices connected by a single edge, etc. In general, after drawing a few paths, it is easy to notice that they all seem to satisfy $e(P_n) = n - 1$, that is, that the following proposition is true, which we will prove by induction.

Proposition B.4.1. *For any $n \geq 1$, the path on n vertices, P_n , has $n - 1$ edges.*

Proof. We want to show this for all paths of length $n \geq 1$ by induction, so we first show it for $n = 1$. That is, that $e(P_1) = 0$. But this is definitely the case, for P_1 is simply one vertex, and a (simple) graph can only have edges between two or more vertices. We now assume that the result holds for all $n \leq k$ and show that the result holds for $n = k + 1$. Consider P_{k+1} , with vertices labeled $v_1, v_2, \dots, v_k, v_{k+1}$, where $v_1v_2, v_2v_3, \dots, v_kv_{k+1} \in E(P_n)$. \square

For another, not too lengthy example of a contradiction proof, see Theorem 1.1.2 in Chapter 1.1. Admittedly, induction is not always this straightforward, and though it always follows this basic idea, it is not an exact mold. For example, at the beginning of Chapter 2.1 we prove a statement about a sequence that is built recursively off two previous terms. In order to prove statements about it for $n \geq 1$, because of this recursive structure, we actually need to prove that the statement we want is true for $n = 1$ and $n = 2$ before we can induct (and prove that it's true in general), and we do so with a slightly varied form. The reader should not get bogged down with this right now, but rather should understand these aforementioned, more basic examples, and then attempt to tackle this idea at the start of Chapter 2.1 when comfortable.

As induction becomes more natural, it becomes easier to see how to bend the mold, and why it makes sense to do so.

We now have enough proof techniques to begin building off of for our purposes, but have only begun to skim the surface of proof theory. For a more comprehensive introduction, see [20].

Appendix C

Introduction to basic combinatorics

Combinatorics is the study of discrete, finite structures. By discrete structures, we mean those made up of isolated parts. Since graphs are, by definition, finite structures made up of separated parts (the vertices), it is easy to see that graph theory is a subfield of combinatorics, and so it is only natural to think that we will need some general combinatorial notions to properly study graph theory. In this appendix, we develop some of these basic combinatorial ideas that will be prerequisites for many sections of this book.

C.1 A brief introduction to enumerative combinatorics

Enumerative combinatorics is an area of combinatorics that deals with the number of ways that certain patterns can be formed. Informally, it is the mathematics of counting, and it shows up all over different areas of combinatorics, with graph theory calling upon its ideas heavily. In this section of this appendix, we develop some of the basic tools of enumerative combinatorics, and look at some associated proofs, which will be combinatorial in nature. That is, will be counting proofs. We begin with the following idea.

The Fundamental Principle of Counting. If there are a ways for one thing to happen, and b ways for another thing to happen, then there are ab ways for them to both happen.

This may sound strange, but it is actually quite intuitive. For example, say you want to order a cheeseburger, and that you have two choices for what type of bun you want, either white or whole wheat. Also, assume that you have three choices for what type of cheese you can put on the burger, either cheddar cheese, mozzarella cheese, or American cheese. How many different burger combinations can you make? The fundamental principle of counting would say that, since there are three choices

of cheese and two choices of bread, that there are $2 \cdot 3 = 6$ possible burgers. We can see that this is true by listing all combinations: white and cheddar, white and mozzarella, white and American, wheat and cheddar, wheat and mozzarella, and wheat and American. But why is this true in general? Well, once you've decided on a type of bread, you have three choices for cheese. So you have three choices of cheese for each type of bread, you have two types of bread, and so you have $3 + 3 = 2 \cdot 3$ choices. Let's say we had five choices of buns. Then, we would have three choices for the first, three choices for the second, etc., so we would have $3 + 3 + 3 + 3 + 3 = 5 \cdot 3$ combinations of burger. The fundamental principle of counting makes sense.

What if, in the above example, in addition to having two choices for type of bun and three choices for type of cheese, we also had three choices for type of meat. How many possibilities burger combinations would we have then? Well, we know that for just the bun and cheese, we have six choices. So, for (the bun and cheese) and (the type of meat) we should have $6 \cdot 3$. That is, the fundamental counting principle should extend to more than two things, and it does. For another example, assume we were running a company that wants to give every employee a four digit ID number, and we decide to require that the first two digits be letters, and the last two digits be numbers between 1 and 9. We can use this idea to easily compute how many employees we can hire before running out of ID numbers. Since there are 26 letters, we have twenty-six possible choices for the first digit and twenty-six choices for the second digit, and we have nine numbers to choose from, so that's nine choices for the third and the fourth digit, so there are $26 \cdot 26 \cdot 9 \cdot 9 = 54,756$ possible ID's.

Consider a similar problem that provides more restriction. Say we want to create 4-digit ID's that are all numbers 1 through 9, but that never have the same number appear next to each other. So, for example, we would want to allow things like 1-8-4-3 but not things like 1-2-2-6. We could frame this as a graph theorem problem. This is equivalent to looking at the graph G defined in example A.2.1 and counting the number of ways to label the vertices with numbers 1 through 9 such that no two numbers which are equal are adjacent. To this end, we could start by labeling v_1 with any number we want. However, once we do, we are not allowed to name v_2 with the same number, for they are adjacent. So we have 9 choices for v_1 , but only 8 for v_2 . Similarly, once we have chosen a number for v_2 , we can't choose the same for v_3 , but v_3 can be anything else. Same for v_4 . So, we have $9 \cdot 8 \cdot 8 \cdot 8 = 4608$ possible ideas of this form. These types of counting proofs take some getting used to, but the examples of the following section should help. Also, the reader can see either [21] for a quite formal introduction to these ideas, or [22] for a more informal, friendly one. We continue to permutations.

In order to develop our next counting tool, we need to define the word *permutation*. Informally, a permutation is an ordered arrangement of given objects. Formally, it is a bijection from a set onto itself. (The formal definition is not necessary for understanding here.) Given the set $\{1, 2, 3\}$ for example, some permutations are 1-2-3, 3-2-1, 2-1-3, etc. Knowing the number of ways to permute a given set of objects is a very helpful counting tool.

Definition. For any non-negative integer n , The *factorial of n* , denoted $n!$, is defined as the number of permutations of any n objects.

Say $n = 3$, that is, that we are finding the number of ordered arrangements of some objects, call them a, b and c . Well, we can have the arrangements: $a - b - c$, $a - c - b$, $b - a - c$, $b - c - a$, $c - a - b$, and $c - b - a$. By listing them out, we see that $3! = 6$. But listing all possible ways is not a practical way of counting in general. What if we wanted to compute, say, $1000!$, or even just $12!$. It turns out that $12!$ is almost 500,000,000. We need an algebraic formula for this quantity. We prove that we have a very nice algebraic form, combinatorially.

Claim C.1.1. $n! := n \cdot (n - 1) \cdots 2 \cdot 1$, the product of all positive integers less than or equal to n .

Proof of claim. We know, by definition, that $n!$ counts the number of arrangements of n objects, and so we need to show that $n \cdot (n - 1) \cdots 2 \cdot 1$ counts the number of arrangements of n objects as well, for this would mean that the quantities are equal. Consider any arbitrary set of n objects, say $\{o_1, o_2, \dots, o_n\}$. We do a count similar to that explained in above. When deciding which object to put first, we can choose any of the n objects. Then (once this object is chosen), we have $n - 1$ choices (any of the elements other than the one we put in the first slot) for which object to put second. Similarly, for which object to put third on the list, we have $n - 2$ choices. We continue this way all the way up until we've placed all but the final object. Since all the others have already been arranged, we have only one choice for this (the one object that's left). Thus, we have n choices for the first object, $n - 1$ for the next, $n - 2$ for the one after, etc. By the fundamental principle of counting, as we did in Example two, we know that the total number of arrangements is the product of these: $n \cdot (n - 1) \cdot (n - 2) \cdots 2 \cdot 1$. The claim is proven. \square

Now we see how much more quickly we could have known that $3! = 3 \cdot 2 \cdot 1 = 6$. A counting tool that will prove more directly useful for our work than the factorial is the binomial coefficient.

Definition. The *binomial coefficient* $\binom{n}{k}$ is defined as the number of k -element subsets of any n -element set, for any positive integers k and n .

It turns out that we can express the binomial coefficient in terms of the factorial, by an identity most commonly referred to as the *factorial formula* for binomial coefficients.

Claim C.1.2 (The Factorial Formula). $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof. We show equivalently that $n! = \binom{n}{k}k!(n-k)!$. Since, we know that $n!$ counts the number of permutations of n objects, to show that $n! = \binom{n}{k}k!(n-k)!$, we need to show that $\binom{n}{k}k!(n-k)!$ counts the number of permutations of n objects as well.

Consider any n objects. We count the number of ways to permute them in a clever way, as follows: We first choose k of the objects (there are $\binom{n}{k}$ ways to do this by definition), then we see how many ways we can arrange those k (there are $k!$); and since the position of the other $(n - k)$ is determined (by the places we didn't put the first k), we need only count the number of arrangements of those with $(n - k)!$ to finish the count. This definitely, also, counts the number of arrangements of n objects. By the fundamental counting principle, since there are $\binom{n}{k}$ ways to choose the k elements, $k!$ ways to permute them, and then $(n - k)!$ ways to permute the remaining objects, there are $\binom{n}{k}k!(n - k)!$ ways to permute n objects, so, by definition, $n! = \binom{n}{k}k!(n - k)!$. This completes the proof. \square

Before closing this section, we quickly define two other related ideas that prove useful. They are, what we call, floors and ceilings. Given integers n and k ($k \neq 0$), we define the *floor* of $\frac{n}{k}$, denoted, $\lfloor \frac{n}{k} \rfloor$, to be the greatest integer which is less than or equal to $\frac{n}{k}$. Similarly, we define the *ceiling* of $\frac{n}{k}$, denoted, $\lceil \frac{n}{k} \rceil$, to be the greatest integer which is less than or equal to $\frac{n}{k}$. So, for example, if n is an even number, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2} = \lceil \frac{n}{2} \rceil$, but if n is odd, then $\lfloor \frac{n-1}{2} \rfloor$ and $\lceil \frac{n+1}{2} \rceil$.

The following section will explain a few of the ways that these counting tools prove useful in graph theory, while also explaining the concept of a subgraph (as mentioned in Chapter 1.1).

C.2 Counting Subgraphs

As defined in Chapter 1.1, a subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Informally, a subgraph is a graph which lies inside another graph. For some examples, let's again refer to example A.2.1 for the final time. If we cover all vertices and edges of the graph except v_2 and v_3 , we see that this forms a P_2 which is a subgraph of our larger P_4 . Similarly, v_1 and v_2 , and v_3 and v_4 also form P_2 subgraphs. We note that this is all possible P_2 subgraphs, as v_1 and v_4 form a graph on two vertices that has no edges (as there are no edges between v_1 and v_4 and H is a graph, as do v_1 and v_3 , and v_2 and v_4). As mentioned in section 1.1, a graph on n vertices that has no edges is called an empty graph, and is denoted E_n .

By the previous paragraph we can say that G has three distinct P_2 subgraphs and three distinct E_2 subgraphs. It turns out that these are all the possible subgraphs of G that contain two vertices. We know this because G contains 4 vertices, and the number of ways to choose 2 vertices from 4 vertices, using Claim C.1.2 is exactly,

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{24}{4} = 6.$$

Since we have counted 6 subgraphs above, it must be all of them.

We will look at and count subgraphs quite a bit throughout this work, using many

different techniques which we will develop. In the same section in which we define subgraphs, we stated that for any graph G and any vertices $v_1, v_2, \dots, v_m \in V(G)$, we define the *subgraph of G induced by vertices v_1, v_2, \dots, v_m* , denoted $G[v_1, v_2, \dots, v_m]$, to be the subgraph of G made up of the vertices v_1, v_2, \dots, v_m and all edges between them. So, when we considered the graph obtained by covering all vertices and edges of the graph except v_2 and v_3 , we were exactly considering the graph which we denote $P_4[v_2, v_3]$.

On the other hand, we introduced notation $G - \{v\}$ to denote the subgraph of G obtained by removing vertex v and all edges incident to v . We then extend this notation to more than one vertex, by, for any vertices $v_1, v_2, \dots, v_m \in V(G)$, letting $G - \{v_1, v_2, \dots, v_m\}$ denote the *subgraph of G obtained by removing vertices v_1, v_2, \dots, v_m and all edges incident to these vertices*. In this example, we can say that $P_4[v_2, v_3]$ is exactly the same graph as $P_4 - \{v_1, v_4\}$.

C.3 Unimodality of binomial coefficients

In Chapter 4.1 we define the concept of unimodality, and use in multiple proofs that sequences of binomial coefficients are unimodal. We prove that assertion here. The reader who has not yet reached the part of this work having to do with unimodality should skip this section until coming across it.

Proposition C.3.1. *For any $l \in \mathbb{N}$, $\left(\binom{l}{t}\right)_{t \in \mathbb{N}}$ is unimodal.*

Proof. We show specifically, that

$$\binom{l}{t} \geq \binom{l}{t-1} \quad \text{if } t \leq \frac{l+1}{2} \quad \text{and} \quad \binom{l}{t} \leq \binom{l}{t-1} \quad \text{if } t \geq \frac{l+1}{2}.$$

Since binomial coefficients are strictly non-negative, it suffices to show that

$$\frac{\binom{l}{t}}{\binom{l}{t-1}} \geq 1 \quad \text{if } t \leq \frac{l+1}{2} \quad \text{and} \quad \frac{\binom{l}{t}}{\binom{l}{t-1}} \leq 1 \quad \text{if } t \geq \frac{l+1}{2}.$$

This is immediate from the definition of a binomial coefficient (as can be found in Appendix C), for using this, we can see that

$$\frac{\binom{l}{t}}{\binom{l}{t-1}} = \frac{\frac{l!}{t!(l-t)!}}{\frac{l!}{(t-1)!(l-t+1)!}} = \frac{l-t+1}{t},$$

which easily satisfies the desired inequalities. □

C.4 A significant result

Consider the following theorem.

Theorem C.4.1. *Any set of n elements has 2^n subsets for any positive integer n .*

This tells us, for example, that if we look at a set with three elements, say $\{a_1, a_2, a_3\}$, then this set should have $2^3 = 8$ subsets. This is true, as

$$\{a_1, a_2, a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}, \{a_1\}, \{a_2\}, \{a_3\} \text{ and } \emptyset,$$

are all the possible subsets of $\{a_1, a_2, a_3\}$. This is not hard to check. We notice here that \emptyset , the empty set, is a subset of any set. This is because, by definition, $\emptyset \subseteq A$ if every element of \emptyset is an element of A , which is of course the case, as \emptyset has no elements. This is a little weird to wrap your mind around at first, but it is a common notion in mathematics.

This theorem will be important to us for a three reasons. First of all, it is a significant, well known result in its own right, and it is one that we reference on multiple occasions throughout different sections of this paper. Second, it will allow us to present a rather complex proof that uses induction, basic set theory and various counting techniques that have been discussed in this section. The third reason is that it will allow us to give an alternate, completely combinatorial proof that is just as valid, but that is a short paragraph long, thus displaying the power and simplicity of such an idea. We have shown that the claim is true when $n = 3$, but again, need to prove it for *all* n . Lets prove this, that any set of n elements has 2^n subsets for any positive integer n , first by the aforementioned induction-style proof.

Proof one. As stated, we need to first show it's true for $n = 1$. That is, we need to show that any set of one element has $2^1 = 2$ subsets. So, consider any arbitrary set consisting of one element, say a . That is, the set $\{a\}$. Then, the only possible subsets of $\{a\}$ are $\{a\}$ itself, and \emptyset , the set with no elements. As noted previously, \emptyset is a subset of any set. We know there can be no other subsets of $\{a\}$ as it only contains that one element, and so we have proven the claim for $n = 1$.

Now we need to prove that if the statement is true for all positive integers less than or equal to any arbitrary positive integer n , then the statement is true for $n + 1$. So, assume that the statement is true for n . That is, assume that every set of n elements has exactly 2^n subsets. Keep in mind throughout that rest of the proof that this is what we are assuming. We need to show that this implies that every set of $k + 1$ elements has exactly 2^{n+1} elements. So, let A be any arbitrary set of n elements. Say, $A = \{a_1, a_2, \dots, a_n, a_{n+1}\}$. We can categorize all subsets of A in a clever way, for example, as the subsets of A that contain the element a_{n+1} and the subsets of A that don't. The subsets of A that do not contain element a_{n+1} are exactly the subsets which use all the other elements, that is, the subsets of $\{a_1, a_2, \dots, a_n\}$. However, $\{a_1, a_2, \dots, a_n\}$ is a set of exactly n elements, and we assumed that every

set of n elements has exactly 2^n subsets, so the number of subsets of A that don't contain a_n is 2^n . Further, every subset of A which contains a_{n+1} must be of the form $\{a_{n+1}\} \cup B$ where $B \subseteq \{a_1, a_2, \dots, a_n\}$. Since B is a subset of $\{a_1, a_2, \dots, a_n\}$, and we have shown that this set has 2^n possible subsets, there are 2^n possible B 's, and thus 2^n possible subsets of A of the form $\{a_{n+1}\} \cup B$ where $B \subseteq \{a_1, a_2, \dots, a_n\}$. Since these are all the possible subsets of A which contain a_{n+1} , we determine that there are 2^n possible subsets of A which contain a_{n+1} . The proof is all but complete, as we can see by grouping what we've done so far together

$$\begin{aligned} \left(\begin{array}{c} \text{Number of subsets} \\ \text{of } A \end{array} \right) &= \left(\begin{array}{c} \text{Number of subsets} \\ \text{that contain } a_{n+1} \end{array} \right) + \left(\begin{array}{c} \text{Number of subsets} \\ \text{that don't contain } a_{n+1} \end{array} \right) \\ &= 2^n + 2^n \\ &= 2(2^n) = 2^{n+1}, \end{aligned}$$

as desired. The claim is proven by mathematical induction. \square

We now offer a combinatorial argument which is much simpler, and which is motivated by the following type of observation. Consider, again, the set $\{a_1, a_2, a_3\}$. If we want to count the possible subsets, we can think of this as having 3 choices to make. Whether or not to include a_1 , whether or not to include a_2 , and whether or not to include a_3 . For example, the subset $\{a_1, a_3\}$ is the subset that does include a_1 , does not include a_2 and does include a_3 . Noting this, this problem boils down to one similar to those discussed when we defined the fundamental principle of counting. Since we have two choices for each element (whether it's included or not), there are $2 \cdot 2 \cdot 2 = 8$ possible subsets. This is 2^3 , as expected. We can use this technique, and provide a beautifully concise proof of Theorem C.4.1, which is considered common knowledge throughout most of advanced mathematics.

Proof two. Let A be a set with n elements. Every subset of A is defined by n choices, each of which has two possibilities, namely, whether or not to include each element. Thus, by the fundamental principle of counting, there are

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}_{n \text{ times}} = 2^n$$

possible subsets. \square